Auslander-Reiten Theory in Triangulated Categories

by

Hongwei Niu

Mémoire présenté au Département de mathématiques
en vue de l'obtention du grade de maîtrise ès sciences

FACULTÉ DES SCIENCES
UNIVERSITÉ DE SHERBROOKE

Sherbrooke, Québec, Canada, juillet 2014

Sommaire

Dans ce mémoire, soit $\mathcal{A}$ une catégorie triangulée et soit $\mathcal{C}$ une sous-catégorie de $\mathcal{A}$ qui est stable pour les extensions. D'abord, nous donnerons quelques nouvelles descriptions de triangles d'Auslander-Reiten dans $\mathcal{C}$. Cela nous donne des conditions nécessaires et suffisantes pour que $\mathcal{C}$ ait des triangles d'Auslander-Reiten. Ensuite, nous étudierons quand un triangle d'Auslander-Reiten dans $\mathcal{A}$ induit un triangle d'Auslander-Reiten dans $\mathcal{C}$. Comme une première application de ces résultats, nous étudierons les triangles d'Auslander-Reiten dans une catégorie triangulée ayant une t-structure. Dans le cas où le cœur de la t-structure est t-héréditaire, nous établirons le lien entre les triangles d'Auslander-Reiten dans $\mathcal{A}$ et les suites d'Auslander-Reiten dans le cœur. Enfin, nous appliquerons nos résultats dans la catégorie dérivée bornée de tous les modules d'une algèbre noethérienne sur un anneau commutatif local noethérien complet. Notre résultat généralise le résultat correspondant de Happel dans la catégorie dérivée bornée des modules de dimension finie d'une algèbre de dimension finie sur un corps algébriquement clos.

Abstract

In this dissertation, let $\mathcal{A}$ be a triangulated category and let $\mathcal{C}$ be an extension-closed subcategory of $\mathcal{A}$. First, we give some new characterizations of an Auslander-Reiten triangle in $\mathcal{C}$, which yields some necessary and sufficient conditions for $\mathcal{C}$ to have Auslander-Reiten triangles. Next, we study when an Auslander-Reiten triangle in $\mathcal{A}$ induces an Auslander-Reiten triangle in $\mathcal{C}$. As an application, we study Auslander-Reiten triangles in a triangulated category with a t-structure. In case the t-structure has a t-hereditary heart, we establish the connection between the Auslander-Reiten triangles in $\mathcal{A}$ and the Auslander-Reiten sequences in the heart. Finally, we specialize to the bounded derived category of all modules of a noetherian algebra over a complete local noetherian commutative ring. Our result generalizes the corresponding result of Happel's in the bounded derived category of finite dimensional modules of a finite dimensional algebra over an algebraically closed field.

Acknowledgements

I would like to express my gratitude to my supervisor Dr. Shiping Liu for his guidance, and my co-supervisor Dr. David Smith for his encouragement, through my master study. I would like also to thank the other professors and colleagues of the research group of representation theory at the Université de Sherbrooke for their help and discussions.

Moreover, I would like to thank Dr. Shiping Liu, Dr. David Smith, the SAG research group, and the Department of Mathematics at the Université de Sherbrooke for the financial support.

Finally, I would like to thank my family for their encouragement and support during my study in Sherbrooke.

Contents

Introduction

The notions of Auslander-Reiten sequences, also known as almost split sequences, was introduced by M. Auslander et I. Reiten in the 1970s. Since then, it has been playing a fundamental role in the representation theory of artin algebras. Later on, M. Auslander et S. O. Smalø developed a theory of Auslander-Reiten sequences in subcategories of abelian categories.

Another later advance was Happel's theory of Aulander-Reiten triangles, which plays the same role in triangulated categories as Auslander-Reiten sequences in abelian categories. It is natural to study when a subcategory of a triangulated category having Auslander-Reiten triangles has Auslander-Reiten triangles. A pioneering work in this direction by P. Jørgensen shows us some results in non-triangulated subcategories of triangulated categories; see [16].

It is a long standing problem to determine which categories have Auslander-Reiten sequences or Auslander-Reiten triangles. Recently, one can characterize an Auslander-Reiten sequence in terms of linear forms on the stable endomorphism algebras of its end terms in an exact category; see [22]. In this dissertation, motivated by the work in [16], [22] and [24], we give some new characterizations of Auslander-Reiten triangles in extension-closed subcategories of a triangulated category, which yield some new existence theorems for Auslander-Reiten triangles in such a subcategory. Finally, we show an application of these results in the bounded derived category of all modules of a noetherian algebra over a complete local noetherian commutative ring.

Next, we introduce more details of this dissertation chapter by chapter.

In Chapter 1, we shall recall some terminology and basic facts on triangulated categories and complete local noetherian rings.

In Chapter 2, we shall characterize an Auslander-Reiten triangle in terms of linear forms on the endomorphism algebras of its end terms. This yields necessary and sufficient conditions for an extension-closed subcategory to have Auslander-Reiten triangles. Specializing to Hom-finite triangulated categories, we recover the result in [24] by I. Reiten and M. Van Den Bergh, who characterized the existence of Auslander-Reiten triangles by the existence of a Serre duality. The result in the second section of Chapter 2, says that if a triangulated category has Auslander-Reiten triangles, then the Auslander-Reiten triangles in a Hom-finite Krull-Schmidt extension-closed subcategory are given by the minimal approximations of the Auslander-Reiten triangle in the ambient category. In fact, this result is due to Jørgensen in [16], but we give a much shorter proof.

In Chapter 3, our main objective is to apply the results obtained in Chapter 2 to triangulated categories with a t-structure. We will give the connection between the Auslander-Reiten triangles in the ambient category and those in the torsion or torsion-free subcategories determined by the t-structure. If the t-structure is bounded with a t-hereditary heart, inspired by the work in [9], we will prove that all the Auslander-Reiten triangles in the triangulated category are the shifts of the Auslander-Reiten sequences in the heart and those of the connecting Auslander-Reiten triangles.

In Chapter 4, we show an application of the results obtained in the previous chapters to the bounded derived category of all modules over a noetherian algebra over a complete local noetherian commutative ring. In the bounded derived category of finite dimensional modules over a finite dimensional algebra over an algebraically closed field, as shown by Happel in [15], the ending terms of Auslander-Reiten triangles are precisely the indecomposable perfect complexes. We will prove that this result also holds in the bounded derived category of all modules over a noetherian algebra. And also, we give the necessary and sufficient conditions when an Auslander-Reiten sequence in the category of all modules over a noetherian algebar induces an Auslander-Reiten triangle in derived category of all modules over a noetherian algebra.

Chapter 1

Preliminaries

In this chapter, we have two objectives: the first one is to recall some basic facts on triangulated categories, and the second one is to recall the notion of a complete local noetherian commutative ring and some basic properties. Considering the context of the dissertation, we shall omit some details. The reader is referred to [12], [13], [23] and [26].

1.1 Triangulated Categories

In this section, we recall the notion of a triangulated category and collect some basic facts in triangulated categories.

Throughout this section, we let $\mathcal{A}$ stand for an additive category. An object in $\mathcal{A}$ is called *strongly indecomposable* if its endomorphism algebra is local. Moreover, a non-zero object is called *Krull-Schmidt* if it is a finite direct sum of strongly indecomposable objects. One says that an idempotent $e : X \to X$ in $\mathcal{A}$ *splits* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{A}$ such that $e = gf$ and $fg = 1_Y$.

Let T be an additive automorphism of $\mathcal{A}$, called *translation functor* on $\mathcal{A}$. If X is an object in $\mathcal{A}$, we shall use the notation $T^n(X) = X[n]$ for all

$n \in \mathbb{N}$. A *triangle* in $\mathcal{A}$ is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A *morphism of triangles* is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1],
\end{tikzcd}$$

which is called an *isomorphism* if f, g, h are isomorphisms.

1.1.1 Definition. An additive category $\mathcal{A}$ is called a *triangulated category* if it is equipped with a translation functor T and a family of triangles, called *exact triangles*, satisfying the following properties:

(1) A triangle isomorphic to an exact triangle is an exact triangle.

(2) For any object X in $\mathcal{A}$, the triangle $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow X[1]$ is an exact triangle.

(3) Each morphism $f : X \to Y$ in $\mathcal{A}$ embeds in an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

(4) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an exact triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is an exact triangle.

(5) Each diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1],
\end{tikzcd}$$

where the rows are exact triangles and the left square is commutative, can be completed to a morphism of exact triangles

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1].
\end{tikzcd}$$

(6) For any morphisms f, g in $\mathcal{A}$, the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[r, "a"] \arrow[d, "g"] & Z' \arrow[r] & X[1] \arrow[d, "1_{X[1]}"] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r, "b"] \arrow[d, "1_Z"] & Y' \arrow[r] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r, "c"] & X' \arrow[r] & Y[1],
\end{tikzcd}$$

where the rows are exact triangles, can be completed to the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[r, "a"] \arrow[d, "g"] & Z' \arrow[r, "r"] \arrow[d, "u"] & X[1] \arrow[d, "1_{X[1]}"] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r, "b"] \arrow[d, "1_Z"] & Y' \arrow[r, "s"] \arrow[d, "v"] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"] \arrow[d, "a"] & Z \arrow[r, "c"] \arrow[d, "b"] & X' \arrow[r, "t"] \arrow[d, "1_{X'}"] & Y[1] \arrow[d, "a[1]"] \\
Z' \arrow[r, "u"] & Y' \arrow[r, "v"] & X' \arrow[r, "w"] & Z'[1],
\end{tikzcd}$$

where all the rows are exact triangles. This property is called the *octahedral axiom.*

For the reader's convenience, here we collect some properties of triangulated categories from [13] and [23], which will be used later.

1.1.2 Lemma ([23]). *Let $\mathcal{A}$ be a triangulated category. If*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is an exact triangle, then $g \circ f = h \circ g = f[1] \circ h = 0$.

1.1.3 Definition. Let $\mathcal{A}$ be a triangulated category and $\mathcal{B}$ be an abelian category. An additive functor $F : \mathcal{A} \to \mathcal{B}$ is called a *cohomological functor* if, for any exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $\mathcal{A}$, we have a long exact sequence

$$\cdots \longrightarrow F(Z[-1]) \xrightarrow{F(h[-1])} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(X[1]) \longrightarrow \cdots$$

in case F is covariant, or a long exact sequence

$$\cdots \longrightarrow F(X[1]) \xrightarrow{F(h)} F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \xrightarrow{F(h[-1])} F(Z[-1]) \longrightarrow \cdots$$

in case F is contravariant.

1.1.4 Proposition ([23]). *Let $\mathcal{A}$ be a triangulated category. For any object U in $\mathcal{A}$, both $\mathrm{Hom}_{\mathcal{A}}(U, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, U)$ are cohomological.*

1.1.5 Proposition ([23]). *Let $\mathcal{A}$ be a triangulated category with an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

(1) *If $u : U \to Z$ is a morphism in $\mathcal{A}$, then $hu = 0$ if and only if $u = gu'$ for some morphism $u' : U \to Y$.*

(2) *If $v : X \to V$ is a morphism in $\mathcal{A}$, then $v \circ h[-1] = 0$ if and only if $v = v'f$ for some morphism $v' : Y \to V$.*

1.1.6 Proposition ([23]). *Let $\mathcal{A}$ be a triangulated category with an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

The following statements are equivalent.

(1) $h = 0$.

(2) f *is a section.*

(3) g *is a retraction.*

1.1.7 Proposition ([23]). *Let $\mathcal{A}$ be a triangulated category with a commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1], \end{tikzcd}$$

where the rows are exact triangles. If any two of the morphisms f, g, h are isomorphisms, then the third one is also an isomorphism.

The following result, which plays the same role as a push-out in abelian categories, is well known; see, for example, the proof of Lemma 2.6 in [17].

1.1.8 Lemma. *Let $\mathcal{A}$ be a triangulated category with an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

For any morphism $\phi : X \to X'$ in $\mathcal{A}$, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] \arrow[d, equal] & X[1] \arrow[d, "\phi[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

with rows being exact triangles; and in any such commutative diagram, $h' = 0$ if and only if ϕ factors through f.

Proof. Let $h' = \phi[1] \circ h$. There exists an exact triangle

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z \xrightarrow{h'} X'[1]$$

which is based on h'. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] \arrow[d, equal] & X[1] \arrow[d, "\phi[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r, "h'"] & X'[1].
\end{tikzcd}$$

It can be completed to a morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi"] & Y \arrow[r, "g"] \arrow[d, "\psi"] & Z \arrow[r, "h"] \arrow[d, equal] & X[1] \arrow[d, "\phi[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z \arrow[r, "h'"] & X'[1].
\end{tikzcd}$$

Moreover, since $h' = \phi[1] \circ h$, we have $h' = 0$ if and only if $\phi \circ h[-1] = 0$, and by Lemma 1.1.5, this is equivalent to have ϕ factoring through f. The proof of the lemma is completed.

Here we state the dual result of Lemma 1.1.8 without a proof.

1.1.9 Lemma. *Let $\mathcal{A}$ be a triangulated category with an exact triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

For each morphism $\phi : Z' \to Z$ in $\mathcal{A}$, there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, equal] & Y' \arrow[r, "g'"] \arrow[d, "\psi"] & Z' \arrow[r, "h'"] \arrow[d, "\phi"] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1],
\end{tikzcd}$$

where the first row is an exact triangle. Moreover, $h' = 0$ if and only if ϕ factors through g.

1.2 Complete Local Notherian Commutative Rings

Throughout this section, we let R be a local commutative ring with a unique maximal ideal $\mathfrak{m}$. Let $(R_n, p_n)_{n\geq 1}$ be a family of R-modules and R-linear maps, where $R_n = R/\mathfrak{m}^n$ and

$$p_n : R_{n+1} \to R_n : x + \mathfrak{m}^{n+1} \mapsto x + \mathfrak{m}^n$$

is the canonical projection.

1.2.1 DEFINITION. The $\mathfrak{m}$-*completion* of R is the submodule of the product $\prod_{n\geq 1} R_n$ defined as follows:

$$\hat{R} = \{(a_1 + \mathfrak{m}, \ldots, a_n + \mathfrak{m}^n, \ldots) \mid p_n(a_{n+1} + \mathfrak{m}^{n+1}) = a_n + \mathfrak{m}^n \text{ for all } n \geq 1\}$$

One says that R is *complete* if the canonical map

$$\sigma : R \to \hat{R} : a \mapsto (a + \mathfrak{m}, \ldots, a + \mathfrak{m}^n, \ldots)$$

is an isomorphism.

EXAMPLE. Let K be a field. The formal power series ring $K[[X]]$ is a complete local noetherian commutative ring.

For the rest of this section, we assume that R is complete local noetherian commutative with maximal ideal $\mathfrak{m}$. Let I be an injective envelope for $R/\mathfrak{m}$. Observe that I is an injective cogenerator of $\mathrm{Mod}R$, the category of all left R-modules. In particular, there exists an exact endofunctor

$$D = \mathrm{Hom}_R(-, I) : \mathrm{Mod}R \to \mathrm{Mod}R.$$

Let A be an R-algebra. One says that A is *semiperfect* if it has a complete set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents such that e_iAe_i is a local ring, for all $1 \leq i \leq n$. Moreover, A is called a *noetherian R-algebra* if A is finitely generated as an R-module. For instance, R is a noetherian algebra over itself. Denote by Mod$A$ the category of all left A-modules, by noe(A) the category of noetherian left A-modules, and by art(A) the category of artinian left A-modules. The following result is well known; see, for example, [12].

1.2.2 PROPOSITION (Matlis). *Let A be a noetherian R-algebra with R being complete local noetherian.*

(1) *A is semiperfect.*

(2) *The categories* noe(A) *and* art(A) *are Krull-Schimidt.*

(3) *The functor $D = \mathrm{Hom}_R(-, I) : \mathrm{Mod}A \to \mathrm{Mod}A^{\mathrm{op}}$ induces a duality, called the* Matlis duality, *as follows*:

$$D : \mathrm{noe}(A) \to \mathrm{art}(A^{\mathrm{op}}).$$

Chapter 2

Auslander-Reiten Theory in Triangulated Categories

Throughout this dissertation, let R stand for a commutative ring. An R-category is a category in which the morphism sets are R-modules and the composition of morphisms is R-bilinear. An R-category will be called *Hom-finite* if all its morphism set are finitely generated as R-modules.

We shall recall briefly the Auslander-Reiten theory from [5, 6]. Let $\mathcal{B}$ be an additive R-category. One says that f is *left almost split* if f is not a section and every non-section morphism $g : X \to L$ factors through f; *left minimal* if any endomorphism $h : Y \to Y$ such that $hf = f$ is an automorphism; and *minimal left almost split* if it is left minimal and left almost split. Dually, we define *right almost split, right minimal* and *minimal right almost split* morphisms. The following result is well known; see [6, (2.3)].

2.0.1. Lemma. *Let $\mathcal{B}$ be an additive R-category and $f : X \to Y$ a morphism in $\mathcal{B}$.*

(1) *If f is left almost split in $\mathcal{B}$, then X is strongly indecomposable.*

(2) *If f is right almost split in $\mathcal{B}$, then Y is strongly indecomposable.*

2.0.2. Definition. *Let $\eta :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be an exact sequence in an abelian category $\mathcal{B}$. One says that η is an Auslander-Reiten sequence if f is left almost split and g is right almost split.*

The following result is probably well known; compare [18, (2.6)]. It slightly generalizes Theorem 2.14 in [7, Section 4].

2.0.3. THEOREM. *Let $\mathcal{B}$ be an abelian R-category with an exact sequence*

$$\eta:\quad 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

If $f : X \to Y$ is minimal left almost split, then η is an Auslander-Reiten sequence.

Proof. Suppose that f is minimal left almost split. Let $\alpha : M \to Z$ be a non-retraction morphism. Pulling-back α along with g, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & N \arrow[r, "v"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0.
\end{tikzcd}$$

Assume that α does not factor through g. This yields that v is not a retraction. Hence, u is not a section. Since f is left almost split, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "\beta'"] & Z \arrow[r] \arrow[d, "\alpha'"] & 0 \\
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & N \arrow[r, "v"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0.
\end{tikzcd}$$

That is, $f = (\beta\beta')f$. Since f is left minimal, $\beta\beta'$ is an automorphism, so is $\alpha\alpha'$. It shows that α is a retraction which is a contradiction. Therefore, g is right almost split, and hence η is an Auslander-Reiten sequence. The proof of the theorem is completed.

2.1 Auslander-Reiten Triangles

In this section, we shall characterize an Auslander-Reiten triangle in terms of linear forms on the endomorphism algebras of its end terms. This yields some sufficient and necessity conditions to have an Auslander-Reiten triangle. There results are analogous to those in [22] for exact categories.

Throughout this section, let $\mathcal{A}$ be a triangulated R-category, and denote by $\mathcal{C}$ an *extension-closed subcategory* of $\mathcal{A}$, that is a full subcategory such that, for any exact triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathcal{A}$, if X, Z are in $\mathcal{C}$, then $Y \in \mathcal{C}$. Note that $\mathcal{C}$ is closed under direct sums, that is, it is additive. By abuse of terminology, an exact triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathcal{A}$ with $X, Y, Z \in \mathcal{C}$ will be called an exact triangle in $\mathcal{C}$.

Now, we state some basic terminology for the Auslander-Reiten theory in an extension-closed subcategory $\mathcal{C}$ of a triangulated R-category $\mathcal{A}$.

2.1.1 Definition. Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$. An exact triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{A}$ is called an *Auslander-Reiten triangle* in $\mathcal{C}$ starting in X and ending in Z if it satisfies the following conditions:

(1) u is left almost split in $\mathcal{C}$; and

(2) v is right almost split in $\mathcal{C}$.

Remark. Even if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$, the object $X[1]$ does not necessarily belong to $\mathcal{C}$.

The following two results are well known; see, for example, [18]. Since they are slightly different from the almost split sequences, we include proofs for self-completeness.

2.1.2 Lemma. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$\eta: \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an exact triangle in $\mathcal{C}$ with $w \neq 0$. If X is strongly indecomposable, then v is right minimal in $\mathcal{C}$.

Proof. Assume that X is strongly indecomposable. Then we get $X[1]$ is also strongly indecomposable. Since η is an Auslander-Reiten triangle, $w \neq 0$. We claim that w is left minimal in $\mathcal{A}$. In fact, let $g : X[1] \to X[1]$ be a morphism in $\mathcal{C}$ such that $gw = w$. Thus, $(1-g)w = 0$. Since $\mathrm{End}_{\mathcal{A}}(X[1])$ is local, $1-g$ is in the radical of $\mathrm{End}_{\mathcal{A}}(X[1])$. Thus, g is an automorphism. The claim is true. Now let $f : Y \to Y$ be a morphism such that $vf = v$. Then we can obtain the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "{h[-1]}"] & Y \arrow[r, "v"] \arrow[d, "f"] & Z \arrow[r, "w"] \arrow[d, equal] & X[1] \arrow[d, "h"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \end{tikzcd}$$

and we get $hw = w$. Hence, h is an isomorphism, so is f. Thus, v is right minimal in $\mathcal{C}$. The proof of the lemma is completed.

2.1.3 Lemma. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$. Let*

$$\eta : \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an exact triangle in $\mathcal{A}$ with $X, Z \in \mathcal{C}$. The following statements are equivalent.

(1) *η is an Auslander-Reiten triangle in $\mathcal{C}$.*

(2) *X is strongly indecomposable and v is right almost split in $\mathcal{C}$.*

(3) *Z is strongly indecomposable and u is left almost split in $\mathcal{C}$.*

Proof. By Lemma 2.0.1, we know that (1) implies (2) and (3). Also, it is clear that (2) and (3) imply (1).

We still need to show that (2) and (3) are equivalent. Suppose that (2) holds. Since v is right almost split, by Lemma 2.0.1, Z is strongly indecomposable. Since X is strongly indecomposable, by Lemma 2.1.2, v is right minimal. Since v is not a retraction, u is not a section. Now suppose that $\phi : X \to X'$ is not a section. By Lemma 1.1.8, we get the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\phi"] & Y \arrow[r, "v"] \arrow[d, "\psi"] & Z \arrow[r, "w"] \arrow[d, equal] & X[1] \arrow[d, "\phi[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z \arrow[r, "w'"] & X'[1]. \end{tikzcd}$$

We claim that u' is a section. If not, v' is not a retraction and factors through v. That is, there exists a morphism $\psi' : Y' \to Y$ such that $v' = v\psi'$. Moreover, we obtain another commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "\phi'"] & Y' \arrow[r, "v'"] \arrow[d, "\psi'"] & Z \arrow[r, "w'"] \arrow[d, equal] & X'[1] \arrow[d, "\phi'[1]"] \\ X \arrow[r, "u"] & Y' \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]. \end{tikzcd}$$

Hence, we get $v = v\psi'\psi$. By minimality, $\psi'\psi$ is an isomorphism. Therefore, $\phi'\phi$ is an isomorphism which implies that ϕ is a section. It is a contradiction to our assumption. Therefore, u' is a section and $w' = 0$. Again, by Lemma 1.1.8, ϕ factors through u and u is left almost split. This shows that (2) implies (3). Dually, we can prove that (3) implies (2). The proof of this lemma is completed.

The following result is stated by Reiten and Van Den Bergh in [24] for Hom-finite triangulated categories. However, we observe that it holds in our general setting with the same proof.

2.1.4 Proposition. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an Auslander-Reiten triangle in $\mathcal{C}$.

(1) *If $w' : L \to X[1]$ with $L \in \mathcal{C}$ is a non-zero morphism in $\mathcal{A}$, then $w = w'f$ for some $f : Z \to L$.*

(2) *If $w' : Z \to L[1]$ is a non-zero morphism in $\mathcal{A}$ with $L \in \mathcal{C}$, then there exists some $g : L \to X$ such that $w = w'g[1]$.*

Proof. We shall prove only (1), since (2) is the dual statement of (1). Assume that $w' : L \to X[1]$ is a non-zero morphism with $L \in \mathcal{C}$. Then we can obtain an exact triangle

$$X \xrightarrow{h} N \longrightarrow L \xrightarrow{w'} X[1]$$

in $\mathcal{A}$, which is based on w'. Since $w' \neq 0$, h is not a section. By the hypothesis, $N \in \mathcal{C}$, and since u is left almost split in $\mathcal{C}$, there exists a morphism $t : Y \to N$ such that $h = tu$. Then we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "t"] & Z \arrow[r, "w"] \arrow[d, "f"] & X[1] \arrow[d, equal] \\ X \arrow[r, "h"] & N \arrow[r] & L \arrow[r, "w'"] & X[1]. \end{tikzcd}$$

Thus, we get f such that $w = w'f$. The proof of the proposition is completed.

In order to strengthen the above results, we shall say that a full subcategory $\mathcal{C}$ of $\mathcal{A}$ is *triangle-stable* provided that, for any exact triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

in $\mathcal{A}$, if any two of X, Y, Z are in $\mathcal{C}$, then so is the third one. Clearly, a triangle-stable subcategory is extension-closed.

The following result is another characterization of an Auslander-Reiten triangle, which extends a result of Reiten and Van den Bergh stated in [24].

2.1.5 Proposition. *Let $\mathcal{C}$ be a triangle-stable subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$\delta : \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an exact triangle in $\mathcal{A}$, where $X, Y, Z \in \mathcal{C}$. The following conditions are equivalent:

(1) *δ is an Auslander-Reiten triangle in $\mathcal{C}$.*

(2) *The object X is strongly indecomposable, and for any non-zero morphism $f : Z \to L$ in $\mathcal{C}$, there exists some morphism $w' : L \to X[1]$ in $\mathcal{A}$ such that $w = w'f$.*

(3) *The object Z is strongly indecomposable, and for any non-zero morphism $w' : L \to X[1]$ in $\mathcal{A}$ with $L \in \mathcal{C}$, there exists some morphism $f : Z \to L$ in $\mathcal{C}$ such that $w = w'f$.*

Proof. First we show that (1) implies (2). First, by Lemma 2.0.1, X is strongly indecomposable. Assume now that $0 \neq f \in \mathrm{Hom}_{\mathcal{C}}(Z, L)$. If f is a section, then there exists a morphism f' such that $f'f = 1_Z$. Setting $w' = wf'$, We have $w'f = wf'f = w$. Otherwise, by the assumption on $\mathcal{C}$, we can obtain an exact triangle

$$(*) \qquad L[-1] \longrightarrow N \xrightarrow{s} Z \xrightarrow{f} L,$$

where $0 \neq N \in \mathcal{C}$, by Proposition 1.1.6. Since f is non-zero, s is not a retraction. Consider the following diagram

$$\begin{tikzcd} & & N \arrow[d, "s"] & \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r, "w"] \arrow[d, "f"] & X[1] \\ & & L & \end{tikzcd}$$

Since s is not a retraction and v is right almost split in $\mathcal{C}$, we have $ws = 0$. Now, by the exactness of the triangle $(*)$, there exists $w' : L \to X[1]$ in $\mathcal{A}$ such that $w = w'f$.

Suppose now that (2) is true. Let $g : M \to Z$ be a non-retraction in $\mathcal{C}$. Then we obtain an exact triangle $M \xrightarrow{g} Z \xrightarrow{f} L \longrightarrow M[1]$, where f is non-zero. By (2), there exists $w' : L \to X[1]$ such that $w = w'f$. By Proposition 1.1.5, $wg = 0$. Thus, δ is an Auslander-Reiten triangle in $\mathcal{C}$.

By Proposition 2.1.4, we know (1) implies (3). Now, suppose (3) holds. Let $g : X \to M$ be a non-section morphism in $\mathcal{C}$. Then we obtain an exact triangle $X \xrightarrow{g} M \longrightarrow L \xrightarrow{w'} X[1]$ with $w' \neq 0$. By (3), there exists $f : Z \to L$ such that $w = w'f$. And we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "h"] & Z \arrow[r, "w"] \arrow[d, "f"] & X[1] \arrow[d, equal] \\ X \arrow[r, "g"] & M \arrow[r] & L \arrow[r, "w'"] & X[1]. \end{tikzcd}$$

That is, $g = hu$. Hence δ is an Auslander-Reiten triangle, by Lemma 2.1.3. The proof of the proposition is completed.

The following statement is a dual result of the preceding result.

2.1.6 PROPOSITION. *Let $\mathcal{C}$ be a triangle-stable subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$\delta: \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an exact triangle in $\mathcal{A}$, where $X, Y, Z \in \mathcal{C}$. The following conditions are equivalent.

(1) *δ is an Auslander-Reiten triangle in $\mathcal{C}$.*

(2) *The object Z is strongly indecomposable, and for any non-zero morphism $g : L \to X$ in $\mathcal{C}$, there exists some morphism $w' : Z[-1] \to L$ in $\mathcal{A}$ such that $w[-1] = gw'$.*

(3) *The object X is strongly indecomposable, and for any non-zero morphism $w' : Z[-1] \to L$ in $\mathcal{A}$ with $L \in \mathcal{C}$, there exists some morphism $g : L \to X$ in $\mathcal{C}$ such that $w[-1] = gw'$.*

As the first application of the preceding result, we obtain the following well-known result, which is stated in [16] in the Hom-finite context.

2.1.7 COROLLARY. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an Auslander-Reiten triangle in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$, as a right module over $\mathrm{End}_{\mathcal{C}}(Z)$, has an essential simple socle generated by w.

Proof. Let S be the submodule of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ generated by w. Let M be an arbitrary submodule of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ with a non-zero element w'. Then by Proposition 2.1.4, there exists a morphism $f : Z \to Z$ such that $w =$

$w'f \in M$. Hence, S is a submodule of M and $S \cap M \neq 0$. By arbitrariness, S is essential submodule of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$. Specializing M to a submodule of S, we get $M = S$. Thus, we know S is the unique and smallest submodule of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$. By the definition of socle, we know S is the essential simple socle of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ as right $\mathrm{End}_{\mathcal{C}}(Z)$-module . The proof of the corollary is completed.

Now we state the dual result without a proof.

2.1.8 Corollary. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an Auslander-Reiten triangle in $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ as a left $\mathrm{End}_{\mathcal{C}}(X)$-module has an essential simple socle generated by w.

From now on, we fix an injective co-generator I for $\mathrm{Mod}R$, the category of all R-modules. Then we have an exact endofunctor

$$D = \mathrm{Hom}_R(-, I) : \mathrm{Mod}R \to \mathrm{Mod}R.$$

2.1.9 Definition. Consider an R-bilinear form

$$< - , - >: U \times V \to I$$

with $U, V \in \mathrm{Mod}R$. It is called *left non-degenerate* provided that, for any non-zero element $u \in U$, there exists some $v \in V$ such that $< u, v > \neq 0$; and *right non-degenerate* provided that, for any non-zero $v \in V$, there exists some $u \in U$ such that $< u, v > \neq 0$; and *non-degenerate* if it is both left and right non-degenerate.

The following result follows immediately from the definition.

2.1.10 Lemma. *Let $< -, - >: U \times V \to I$ be an R-bilinear form with $U, V \in \mathrm{Mod}R$.*

(1) *If* $<-,->$ *is left non-degenerate, then there exists a monomorphism*

$$\Phi_U : U \to DV : u \mapsto < u, - > .$$

(2) *If* $<-,->$ *is right non-degenerate, then there exists a monomorphism*

$$\Phi_V : V \to DU : v \mapsto < -, v > .$$

Let $X, Z \in \mathcal{C}$. Every R-linear form $\phi : \mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \to I$ determines, for each $L \in \mathcal{C}$, an R-bilinear form

$$< - , - >_\phi : \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \times \mathrm{Hom}_{\mathcal{C}}(Z, L) \to I : (f, g) \mapsto \phi(fg);$$

and an R-bilinear form

$${}_\phi < - , - > : \mathrm{Hom}_{\mathcal{C}}(L, X) \times \mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \to I : (f, g) \mapsto \phi((fg)[1]).$$

Observe that, for each non-zero morphism $h \in \mathrm{Hom}_{\mathcal{A}}(Z, X)$, since I is an injective co-generator, there exists an R-linear form $\phi : \mathrm{Hom}_{\mathcal{A}}(Z, X) \to I$ such that $\phi(h) \neq 0$.

2.1.11 LEMMA. *Let* $\mathcal{C}$ *be an extension-closed subcategory of a triangulated* R*-category* $\mathcal{A}$*, which admits an Auslander-Reiten triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

Fix an R*-linear form* $\phi : \mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \to I$ *such that* $\phi(w) \neq 0$.

(1) *For any* $L \in \mathcal{C}$*, the* R*-bilinear form*

$$< - , - >_\phi : \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \times \mathrm{Hom}_{\mathcal{C}}(Z, L) \to I : (f, g) \mapsto \phi(fg)$$

is left non-degenerate.

(2) *For every* $L \in \mathcal{C}$*, the* R*-bilinear form*

$${}_\phi < - , - > : \mathrm{Hom}_{\mathcal{C}}(L, X) \times \mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \to I : (f, g) \mapsto \phi((fg)[1])$$

is right non-degenerate.

(3) *If $\mathcal{C}$ is a triangle-stable subcategory of $\mathcal{A}$, then both $< -, - >_\phi$ and ${}_\phi < -, - >$ are non-degenerate for any $L \in \mathcal{C}$.*

Proof. We shall prove the lemma only for the first R-bilinear form $< -, - >_\phi$. Fix an object $L \in \mathcal{C}$. Let $f \in \mathrm{Hom}_{\mathcal{A}}(L, X[1])$ be a non-zero morphism. By Proposition 2.1.4, there exists a morphism $g : Z \to L$ such that $w = fg$. Obviously, $< f, g >_\phi = \phi(fg) = \phi(w) \neq 0$. This shows that $< -, - >_\phi$ is left non-degenerate.

Assume that $\mathcal{C}$ is triangle-stable. Let $0 \neq g \in \mathrm{Hom}_{\mathcal{C}}(Z, L)$. By Proposition 2.1.5, there exists a morphism $f : L \to X[1]$ such that $w = fg$. As a consequence, $< f, g >_\phi = \phi(fg) = \phi(w) \neq 0$. That is, $< -, - >_\phi$ is right non-degenerate. The proof of the lemma is completed.

The following statements are analogous to the results of Liu, Ng and Paquette for exact subcategories of abelian categories; see [22].

2.1.12 Definition. Let $F : \mathcal{A} \to \mathrm{Mod}R$ and $G : \mathcal{A} \to \mathrm{Mod}R$ be covariant R-linear functors. A *functorial monomorphism* $\varphi : F \to G$ is a natural transformation such that $\varphi_X : F(X) \to G(X)$ is a monomorphism for all $X \in \mathcal{A}$.

2.1.13 Definition. Let Λ be an R-algebra. A non-zero R-linear form $\varphi : \Lambda \to I$ is called *almost vanishing* if it vanishes on the Jacobson radical of Λ.

2.1.14 Theorem. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let*

$$\eta : \quad X \longrightarrow Y \longrightarrow Z \xrightarrow{w} X[1]$$

be an exact triangle in $\mathcal{A}$, where $X, Z \in \mathcal{C}$ are strongly indecomposable. Then the following statements are equivalent.

(1) *The exact triangle η is an Auslander-Reiten triangle in $\mathcal{C}$.*

(2) *There exists a functorial monomorphism*

$$\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(Z, -)$$

such that $\varphi_Z(w)$ is almost vanishing on $\mathrm{End}_{\mathcal{C}}(Z)$.

(3) *There exists a functorial monomorphism*

$$\psi : \mathrm{Hom}_{\mathcal{A}}(Z[-1], -)|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(-, X)$$

such that $\psi_X(w[-1])$ *is almost vanishing on* $\mathrm{End}_{\mathcal{C}}(X)$.

Proof. We shall only prove the equivalence of Statement (1) and (2). Assume that η is an Auslander-Reiten triangle in $\mathcal{C}$. Since $w \neq 0$, there exists an R-linear form $\phi : \mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \to I$ such that $\phi(w) \neq 0$. Fix $L \in \mathcal{C}$. By Lemma 2.1.11, we have a left non-degenerate R-bilinear form

$$< -, - >_{\phi} : \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \times \mathrm{Hom}_{\mathcal{C}}(Z, L) \to I : (f, g) \mapsto \phi(fg).$$

By Lemma 2.1.10, this induces an R-linear monomorphism

$$\varphi_L : \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \to D\mathrm{Hom}_{\mathcal{C}}(Z, L) : f \mapsto < f, - >_{\phi} .$$

We claim that φ is natural in L. Let $g : N \to L$ be a morphism in $\mathcal{C}$. Consider the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(L, X[1]) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(g, X[1])"] \arrow[d, "\varphi_L"'] & \mathrm{Hom}_{\mathcal{A}}(N, X[1]) \arrow[d, "\varphi_N"] \\
D\mathrm{Hom}_{\mathcal{C}}(Z, L) \arrow[r, "D\mathrm{Hom}_{\mathcal{C}}(Z, g)"] & D\mathrm{Hom}_{\mathcal{C}}(Z, N).
\end{tikzcd}$$

Let $f \in \mathrm{Hom}_{\mathcal{A}}(L, X[1])$. For any $h \in \mathrm{Hom}_{\mathcal{C}}(Z, N)$, we have

$$(D\mathrm{Hom}_{\mathcal{A}}(Z, g) \circ \varphi_L(f))(h) = (\varphi_L(f))(gh) =< f, gh >_{\phi} = \phi(fgh).$$

On the other hand,

$$(\varphi_N \circ (\mathrm{Hom}_{\mathcal{A}}(g, X[1])(f)))(h) = (\varphi_N(fg))(h) =< fg, h >_{\phi} = \phi(fgh).$$

Thus, the above diagram commutes, that is, the claim is true.

Since φ_Z is injective, $\varphi_Z(w) \neq 0$. If $f \in \mathrm{rad}(\mathrm{End}_{\mathcal{C}}(Z))$, then f is not a retraction. By the properties of Auslander-Reiten triangles, we have $wf = 0$, and therefore,

$$\varphi_Z(w)(f) =< w, f >_{\phi} = \phi(wf) = 0.$$

Thus, $\varphi_Z(w)$ is almost vanishing on $\mathrm{End}_{\mathcal{C}}(Z)$.

Conversely, let $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(Z, -)$ be a functorial monomorphism such that $\varphi_Z(w)$ is almost vanishing on $\mathrm{End}_{\mathcal{C}}(Z)$. In particular, $\varphi_Z(w) \neq 0$, and thus, $w \neq 0$. Let $u : L \to Z$ is a morphism in $\mathcal{C}$ which is not a retraction. For any morphism $v : Z \to L$, since $\mathrm{End}_{\mathcal{C}}(Z)$ is local, $uv \in \mathrm{rad}(\mathrm{End}_{\mathcal{C}}(Z))$. Thus $\varphi_Z(w)(uv) = 0$, that is, $D\mathrm{Hom}_{\mathcal{C}}(Z, u) \circ \varphi_Z(w) = 0$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \arrow[r, "\mathrm{Hom}_{\mathcal{A}}(u, X[1])"] \arrow[d, "\varphi_Z"'] & \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \arrow[d, "\varphi_L"] \\
D\mathrm{Hom}_{\mathcal{C}}(Z, Z) \arrow[r, "D\mathrm{Hom}_{\mathcal{C}}(Z, u)"] & D\mathrm{Hom}_{\mathcal{C}}(Z, L),
\end{tikzcd}$$

we have $(\varphi_L \circ \mathrm{Hom}_{\mathcal{C}}(u, X[1]))(w) = (\mathrm{Hom}_{\mathcal{C}}(Z, u) \circ \varphi_Z)(w) = 0$. Since φ_L is injective, $wu = \mathrm{Hom}_{\mathcal{A}}(u, X[1])(w) = 0$. Hence,

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{w} X[1]$$

is an Auslander-Reiten triangle in $\mathcal{C}$. The proof of the theorem is completed.

If $X, Y \in \mathcal{A}$, then $D\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is an $\mathrm{End}_{\mathcal{A}}(X)$-$\mathrm{End}_{\mathcal{A}}(Y)$-bimodule with multiplications defined, for $f \in \mathrm{End}_{\mathcal{A}}(X), \theta \in D\mathrm{Hom}_{\mathcal{A}}(X, Y), g \in \mathrm{End}_{\mathcal{A}}(Y)$, by

$$f\theta g : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to I : h \mapsto \theta(ghf).$$

The following result is an existence theorem for Auslander-Reiten triangles.

2.1.15 Theorem. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$. If X, Z are strongly indecomposable objects in $\mathcal{C}$, then the following statements are equivalent.*

(1) *There exists an Auslander-Reiten triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathcal{C}$.*

(2) *The $\mathrm{End}_{\mathcal{C}}(Z)$-socle of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ is non-zero, and there exists a functorial monomorphism $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(Z, -)$.*

(3) *The* $\mathrm{End}_{\mathcal{C}}(X)$*-socle of* $\mathrm{Hom}_{\mathcal{A}}(Z[-1], X)$ *is non-zero, and there exists a functorial monomorphism* $\phi : \mathrm{Hom}_{\mathcal{A}}(Z[-1], -)|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(-, X)$.

Proof. We shall only prove the equivalence of (1) and (2). First assume that $\mathcal{C}$ has an Auslander-Reiten triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

Thus, $w \neq 0$. By the previous theorem, we have a functorial monomorphism $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{A}}(Z, -)$. By Lemma 2.1.7, we know w is a non-zero element in the $\mathrm{End}_{\mathcal{C}}(Z)$-socle of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$.

Conversely, assume that $w : Z \to X[1]$ is a non-zero element in the $\mathrm{End}_{\mathcal{C}}(Z)$-socle of $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$. Then we can get an exact triangle

$$\eta : \quad X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

Let $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \to D\mathrm{Hom}_{\mathcal{C}}(Z, -)$ be a functorial monomorphism. Since φ is natural, $\varphi_Z : \mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \to D\mathrm{Hom}_{\mathcal{C}}(Z, Z)$ is $\mathrm{End}_{\mathcal{C}}(Z)$-linear. Hence, $\varphi_Z(w)$ is a non-zero element of $D\mathrm{End}_{\mathcal{C}}(Z)$. If $g \in \mathrm{rad}(\mathrm{End}_{\mathcal{C}}(Z))$, then $(\varphi_Z(w))(g) = \varphi(wg) = 0$. Thus, $\varphi_Z(w)$ is almost vanishing on $\mathrm{End}_{\mathcal{C}}(Z)$. By Theorem 2.1.14, η is an Auslander-Reiten triangle in $\mathcal{C}$. The proof of the theorem is completed.

As another application of Theorem 2.1.14, we have the following sufficiency condition for the existence of Auslander-Reiten triangles in an extension-closed subcategory of a triangulated category.

2.1.16 Proposition. *Let* $\mathcal{C}$ *be an extension-closed subcategory of a triangulated* R*-category* $\mathcal{A}$*, and let* X, Z *be strongly indecomposable objects in* $\mathcal{C}$*. If there exists a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \cong D\mathrm{Hom}_{\mathcal{C}}(Z, -) \text{ or } \mathrm{Hom}_{\mathcal{A}}(Z[-1], -)|_{\mathcal{C}} \cong D\mathrm{Hom}_{\mathcal{C}}(-, X),$$

then there exists an Auslander-Reiten triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ *in* $\mathcal{C}$*.*

Proof. We assume that $\varphi : \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}} \cong D\mathrm{Hom}_{\mathcal{C}}(Z, -)$ is a functorial isomorphism. In particular, $\varphi_Z : \mathrm{Hom}_{\mathcal{A}}(Z, X[1]) \to D\mathrm{End}_{\mathcal{C}}(Z)$ is an isomorphism. Let $p : \mathrm{End}_{\mathcal{C}}(Z) \to \mathrm{End}_{\mathcal{C}}(Z)/\mathrm{rad}(\mathrm{End}_{\mathcal{C}}(Z))$ be the canonical projection. Since I is an injective cogenerator, there exists a non-zero morphism $\theta : \mathrm{End}_{\mathcal{C}}(Z)/\mathrm{rad}(\mathrm{End}_{\mathcal{C}}(Z)) \to I$. Setting $\phi = \theta p$, it is easy to see that ϕ is almost vanishing on $\mathrm{End}_{\mathcal{C}}(Z)$. Since φ_Z is an isomorphism, there exists $w \in \mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ such that $\varphi_Z(w) = \phi$. By Theorem 2.1.14, we obtain an Auslander-Reiten triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$. The proof of the proposition is completed.

Next, we shall study when the sufficient condition stated in the above proposition is also necessary.

The following result is well known. Here, we give a proof for self-completeness.

2.1.17 Lemma. *Let R be a complete local noetherian commutative ring. Let $U, V \in \mathrm{Mod}R$ and consider an R-bilinear form*

$$< -, - >: U \times V \to I.$$

(1) *If $< -, - >$ is left non-degenerate and V is noetherian or artinian, then*

$$\Phi_V : V \to DU : v \mapsto < -, v >$$

is an epimorphism.

(2) *If $< -, - >$ is right non-degenerate and U noetherian or artinian, then,*

$$\Phi_U : U \to DV : u \mapsto < u, - >$$

is an epimorphism.

Proof. We shall prove only (1). Suppose V is noetherian or artinian. By Proposition 1.2.2, there exists an isomorphism

$$\Psi : V \to D^2V : v \mapsto \Psi_v,$$

where $\Psi_v : DV \to I : \psi \to \psi(v)$. Suppose that $< -, - >$ is left non-degenerate. By Lemma 2.1.10, we have a monomorphism

$$\Phi_U : U \to DV : \; u \mapsto \varphi_u := < u, - > .$$

Applying the duality D, we obtain an epimorphism

$$D(\Phi_U) : D^2V \to DU : \theta \mapsto \theta \cdot \Phi_U.$$

Hence, for each $\phi \in DU$, there exists some $v \in V$ such that

$$\phi = D(\Phi)(\Psi(v)) = \Psi_v \Phi_U.$$

For any $u \in U$, we have

$$\phi(u) = (\Psi_v \Phi_U)(u) = \Psi_v(\varphi_u) = \varphi_u(v) =< u, v > .$$

That is, $\phi =< -, v >$. The proof of the lemma is completed.

2.1.18 Corollary. *Let R be a complete local noetherian commutative ring. Consider a non-degenerate R-bilinear form*

$$< -, - >: U \times V \to I,$$

where U, V are R-modules. If one of U and V is noetherian or artinian, then

$$\Phi_V : V \to DU : v \mapsto < - , v > \text{ and } \Phi_U : U \to DV : u \mapsto < - , u >$$

are R-linear isomorphisms.

Proof. Since the R-bilinear form $< -, - >$ is non-degenerate, by Lemma 2.1.10, both Φ_V and Φ_U are monomorphisms. If one of U and V is noetherian or artinian, then so is the other one. Now the result follows from Lemma 2.1.17. The proof of the corollary is completed.

2.1.19 Theorem. *Let $\mathcal{C}$ be a triangle-stable subcategory of a triangulated R-category $\mathcal{A}$, where R is complete local noetherian. Let $\mathcal{C}$ have an Auslander-Reiten triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ and an object $L \in \mathcal{C}$.*

(1) *If one of* $\mathrm{Hom}_{\mathcal{A}}(L, X[1])$ *and* $\mathrm{Hom}_{\mathcal{C}}(Z, L)$ *is noetherian or artinian, then*

$$\mathrm{Hom}_{\mathcal{A}}(L, X[1]) \cong D\mathrm{Hom}_{\mathcal{C}}(Z, L).$$

(2) *If one of* $\mathrm{Hom}_{\mathcal{A}}(Z[-1], L)$ *and* $\mathrm{Hom}_{\mathcal{C}}(L, X)$ *is noetherian or artinian, then*
$$\mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \cong D\mathrm{Hom}_{\mathcal{C}}(L, X).$$

Proof. We shall only prove Statement (1). By Lemma 2.1.11, there exists a non-degenerate R-bilinear form

$$< -, - >: \mathrm{Hom}_{\mathcal{A}}(L, X[1]) \times \mathrm{Hom}_{\mathcal{C}}(Z, L) \to I.$$

If one of $\mathrm{Hom}_{\mathcal{A}}(L, X[1])$ and $\mathrm{Hom}_{\mathcal{C}}(Z, L)$ is noetherian or artinian, by Corollary 2.1.18, we get $\mathrm{Hom}_{\mathcal{A}}(L, X[1]) \cong D\mathrm{Hom}_{\mathcal{C}}(Z, L)$. The proof of this theorem is completed.

The following result generalizes slightly the well-known result by Reiten and Van den Bergh stated in [24].

2.1.20 Theorem. *Let* $\mathcal{C}$ *be a Hom-finite triangle-stable subcategory of a triangulated* R*-category* $\mathcal{A}$*, where* R *is a complete local noetherian ring. If* X, Z *are strongly indecomposable objects in* $\mathcal{C}$*, then the following conditions are equivalent.*

(1) $\mathcal{C}$ *has an Auslander-Reiten triangle* $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$.

(2) *There exists an isomorphism* $D\mathrm{Hom}_{\mathcal{C}}(Z, -) \cong \mathrm{Hom}_{\mathcal{A}}(-, X[1])|_{\mathcal{C}}$.

(3) *There exists an isomorphism* $D\mathrm{Hom}_{\mathcal{C}}(-, X) \cong \mathrm{Hom}_{\mathcal{A}}(Z[-1], -)|_{\mathcal{C}}$.

Proof. We only prove that (1) and (2) are equivalent. We can get that (2) implies (1) from Proposition 2.1.16. Conversely since $\mathrm{Hom}_{\mathcal{C}}(Z, L)$ is Hom-finite, for all $L \in \mathcal{C}$, we know that $\mathrm{Hom}_{\mathcal{C}}(Z, L)$ is noetherian. By Theorem 2.1.19, we know that (1) implies (2). The proof of the corollary is completed.

2.2 Minimal Approximations

Throughout this section, $\mathcal{A}$ stands for a triangulated R-category, where R is a commutative ring, and $\mathcal{C}$ stands for an extension-closed subcategory of $\mathcal{A}$. In this section, our objective is to study when an Auslander-Reiten triangle in $\mathcal{A}$ induces an Auslander-Reiten triangle in $\mathcal{C}$.

Now we introduce some terminology.

2.2.1 DEFINITION. Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let X be an object in $\mathcal{A}$.

(1) A morphism $f : M \to X$ in $\mathcal{A}$ with $M \in \mathcal{C}$ is called a *right $\mathcal{C}$-approximation* of X if the map $\mathrm{Hom}_{\mathcal{A}}(L, f) : \mathrm{Hom}_{\mathcal{C}}(L, M) \to \mathrm{Hom}_{\mathcal{A}}(L, X)$ is surjective for any $L \in \mathcal{C}$.

(2) A morphism $g : X \to N \in \mathcal{A}$ with $N \in \mathcal{C}$ is called a *left $\mathcal{C}$-approximation* of X if the map $\mathrm{Hom}_{\mathcal{A}}(g, L) : \mathrm{Hom}_{\mathcal{C}}(N, L) \to \mathrm{Hom}_{\mathcal{A}}(X, L)$ is surjective for any $L \in \mathcal{C}$.

2.2.2 DEFINITION. Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let X be an object in $\mathcal{A}$.

(1) A morphism $f : M \to X$ in $\mathcal{A}$ with $M \in \mathcal{C}$ is called a *minimal right $\mathcal{C}$-approximation* of X if f is right minimal and a right $\mathcal{C}$-approximation of X.

(2) A morphism $f : X \to M$ in $\mathcal{A}$ with $M \in \mathcal{C}$ is called a *minimal left $\mathcal{C}$-approximation* of X if f is left minimal and a left $\mathcal{C}$-approximation of X.

The following result is stated by Jørgensen in [16] in the Hom-finite context. However, we observe that it holds in the general context.

2.2.3 LEMMA. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ be an Auslander-Reiten triangle in $\mathcal{A}$. Suppose that $Z \in \mathcal{C}$ and $f : M \to X$ is a minimal right $\mathcal{C}$-approximation of X.*

(1) *For $L \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{A}}(L, f[1]) : \mathrm{Hom}_{\mathcal{A}}(L, M[1]) \to \mathrm{Hom}_{\mathcal{A}}(L, X[1])$ is a monomorphism.*

(2) *M is zero or indecomposable.*

Proof. (1) Suppose that there exists a morphism $g \in \mathrm{Hom}_{\mathcal{A}}(L, M[1])$ such that $f[1] \circ g = 0$. Then $\mathcal{A}$ has two exact triangles

$$M \xrightarrow{r} E \longrightarrow L \xrightarrow{g} M[1]$$

and

$$X \xrightarrow{s} F \longrightarrow L \xrightarrow{0} X[1].$$

where the first one is based on g and the second one is based on $0 : L \to X[1]$. Since $f[1] \circ g = 0$, we can get a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "r"] \arrow[d, "f"] & E \arrow[r] \arrow[d, "h"] & L \arrow[r, "g"] \arrow[d, equal] & M[1] \arrow[d, "f[1]"] \\
X \arrow[r, "s"] & F \arrow[r] & L \arrow[r, "0"] & X[1].
\end{tikzcd}$$

Obviously, s is a section. Hence there exists $t : F \to X$ such that $ts = 1_X$ and $f = (th)r$. Since $E \in \mathcal{C}$ and f is a right approximation of X, there exists $u : E \to M$ such that $fu = th$. Then, we know $f = (th)r = fur$. Since f is right minimal, ur is an isomorphism and r is a section. Therefore, $g = 0$. That is, $\mathrm{Hom}_{\mathcal{A}}(L, f[1])$ is injective.

(2) By (1), we know that

$$\mathrm{Hom}_{\mathcal{A}}(Z, f[1]) : \mathrm{Hom}_{\mathcal{A}}(Z, M[1]) \to \mathrm{Hom}_{\mathcal{A}}(Z, X[1])$$

is injective. By Corollary 2.1.7, $\mathrm{Hom}_{\mathcal{A}}(Z, X[1])$ has an essential simple socle as an $\mathrm{End}(Z)_{\mathcal{C}}$-module, and thus, each of its $\mathrm{End}(Z)_{\mathcal{C}}$-submodules is indecomposable. In particular, $\mathrm{Hom}_{\mathcal{A}}(Z, M[1])$ is indecomposable.

Assume now that $M = M_1 \oplus M_2$, with non-zero canonical injections $q_i : M_1 \to M$, $i = 1, 2$. Setting $f_i = fq_i$, since $f : M \to X$ is right minimal, $f_i \neq 0$, for $i = 1, 2$. Now we claim that $\mathrm{Hom}_{\mathcal{A}}(Z, M_i[1]) \neq 0$, for $1 = 1, 2$. In fact, applying Lemma 2.1.11(3) to the Auslander-Reiten triangle in $\mathcal{A}$ stated in the lemma, we obtain a non-degenerate R-bilinear form

$$\mathrm{Hom}_{\mathcal{A}}(M_i, X) \times \mathrm{Hom}_{\mathcal{A}}(Z[-1], M_i) \to I.$$

Since $0 \neq f_i \in \mathrm{Hom}_{\mathcal{A}}(M_i, X)$, we know $\mathrm{Hom}_{\mathcal{A}}(Z[-1], M_i) \neq 0$, that is, $\mathrm{Hom}_{\mathcal{A}}(Z, M_i[1]) \neq 0$, for $i = 1, 2$. Therefore,

$$\mathrm{Hom}_{\mathcal{A}}(Z, M[1]) = \mathrm{Hom}_{\mathcal{A}}(Z, M_1[1]) \oplus \mathrm{Hom}_{\mathcal{A}}(Z, M_2[1])$$

is decomposable, a contradiction. The proof of the lemma is completed.

2.2.4 Definition. Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated category $\mathcal{A}$. An object X in $\mathcal{C}$ is called *Ext-projective* in $\mathcal{C}$ provided that $\mathrm{Hom}_{\mathcal{A}}(X, Y[1]) = 0$, for any $Y \in \mathcal{C}$. Dually, X is called *Ext-injective* in $\mathcal{C}$ provided that $\mathrm{Hom}_{\mathcal{A}}(Y, X[1]) = 0$, for any $Y \in \mathcal{C}$.

Remark. If $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$, then Z is not Ext-projective and X is not Ext-injective in $\mathcal{C}$.

The following result is also stated by Jørgensen in [16] in the Hom-finite context. Our statement is a little more general with a much shorter proof.

2.2.5 Proposition. *Let $\mathcal{C}$ be an extension-closed subcategory of a triangulated R-category $\mathcal{A}$, and let Z be an object in $\mathcal{C}$, which admits an Auslander-Reiten triangle $X \longrightarrow Y \longrightarrow Z \xrightarrow{w} X[1]$ in $\mathcal{A}$. Suppose that X has a minimal right $\mathcal{C}$-approximation $f : M \to X$.*

(1) *The object Z is not Ext-projective in $\mathcal{C}$ if and only if $f \neq 0$.*

(2) *If Z is not Ext-projective in $\mathcal{C}$, then we have a commutative diagram*

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & M[1] \arrow[d, "f[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1], \end{tikzcd}$$

where the rows are exact. If, moreover, M is Krull-Schmidt, then the upper row in any such commutative diagram is an Auslander-Reiten triangle in $\mathcal{C}$.

Proof. (1) Assume that $f \neq 0$. By the non-degeneracy stated in Lemma 2.1.11(3), $\mathrm{Hom}_{\mathcal{A}}(Z, M[1]) \neq 0$. Hence, Z is not Ext-projective in $\mathcal{C}$. Conversely, assume that Z is not Ext-projective in $\mathcal{C}$. That is, there exists an object $L \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{A}}(Z, L[1]) \neq 0$. By Lemma 2.1.11(3), there exists a non-zero element $h \in \mathrm{Hom}_{\mathcal{A}}(L, X) \neq 0$. Since f is a minimal right $\mathcal{C}$-approximation, there exists $g : M \to L$ such that $h = fg$. Hence, $f \neq 0$.

(2) Assume that Z is not Ext-projective, by (1), we know $f \neq 0$. Then by Proposition 2.1.6, there exists a morphism $g : Z \to M[1]$ such that $w = f[1]g$. Thus, there exists a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & N \arrow[r, "t"] \arrow[d, dotted] & Z \arrow[r, "g"] \arrow[d, equal] & M[1] \arrow[d, "f[1]"] \\ X \arrow[r] & Y \arrow[r] & Z \arrow[r, "w"] & X[1], \end{tikzcd}$$

where the upper row is an exact triangle based on g with $M, N \in \mathcal{C}$.

Assume now M is Krull-Schmidt. By Lemma 2.2.3, M is strongly indecomposable. Let $h : L \to Z$ be a non-retraction morphism in $\mathcal{C}$. Then $wh = 0$, and hence, $f[1]gh = 0$. By Lemma 2.2.3, we know that the map

$$\mathrm{Hom}_{\mathcal{A}}(L, f[1]) : \mathrm{Hom}_{\mathcal{A}}(L, M[1]) \to \mathrm{Hom}_{\mathcal{A}}(L, X[1])$$

is a monomorphism. Thus, we get $gh = 0$. By Proposition 1.1.5, h factors through t. By Lemma 2.1.3, $M \longrightarrow N \xrightarrow{t} Z \xrightarrow{g} M[1]$ is an Auslander-Reiten triangle in $\mathcal{C}$. The proof of the proposition is completed.

The following result extends slightly a result of Jørgensen stated in [16]. We give a complete proof for self-completeness.

2.2.6 Theorem. *Let R be a complete local noetherian ring. Let*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

be an Auslander-Reiten triangle in $\mathcal{A}$, and $M \xrightarrow{r} N \xrightarrow{t} Z \xrightarrow{s} M[1]$ be an Auslander-Reiten triangle in $\mathcal{C}$. If $\mathrm{Hom}_{\mathcal{C}}(L, M) \in \mathrm{mod}R$ for any $L \in \mathcal{C}$, then there exists a commutative diagram

$$\begin{tikzcd} M \arrow[r, "r"] \arrow[d, "f"] & N \arrow[r, "t"] \arrow[d, "g"] & Z \arrow[r, "s"] \arrow[d, equal] & M[1] \arrow[d, "f[1]"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1], \end{tikzcd}$$

where f is a minimal $\mathcal{C}$-approximation of X and g is a right $\mathcal{C}$-approximation of Y.

Proof. Since $s \neq 0$, by Proposition 2.1.5, we know there exists a morphism $f : M \to X$ such that $w = f[1]s$. Hence, we establish a commutative diagram as follows:

$$\begin{tikzcd} M \arrow[r, "r"] \arrow[d, "f"] & N \arrow[r, "t"] \arrow[d, "g"] & Z \arrow[r, "s"] \arrow[d, equal] & M[1] \arrow[d, "f[1]"] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]. \end{tikzcd}$$

We claim that $f : M \to X$ is a right $\mathcal{C}$-approximation of X. Indeed, let $f_0 : L \to X$ be a non-zero morphism in $\mathcal{A}$ with $L \in \mathcal{C}$. Fix an R-linear form $\varphi : \mathrm{Hom}_{\mathcal{A}}(Z[-1], X) \to I$ such that $\varphi(w[-1]) \neq 0$. By Lemma 2.1.11(3), we have a non-degenerate R-bilinear form

$${}_{\varphi} < -, - >: \mathrm{Hom}_{\mathcal{A}}(L, X) \times \mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \to I : (q, i) \mapsto \varphi(qi).$$

Now, consider the R-linear form

$$\psi : \mathrm{Hom}_{\mathcal{A}}(Z[-1], M) \to I : u \mapsto \varphi(fu),$$

which is such that $\psi(s[-1]) = \varphi(fs[-1]) = \varphi(w[-1]) \neq 0$. By Lemma 2.1.11(2), we have a right non-degenerate R-bilinear form

$${}_{\psi} < -, - >: \mathrm{Hom}_{\mathcal{C}}(L, M) \times \mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \to I : (p, j) \mapsto \psi(pj).$$

Using $f_0 : L \to X$, we obtain a R-linear map

$${}_{\varphi}< f_0, - >: \mathrm{Hom}_{\mathcal{A}}(Z[-1], L) \to I : i \mapsto {}_{\varphi}< f_0, i > .$$

Suppose that $\mathrm{Hom}_{\mathcal{C}}(L, M)$ is finitely generated, and hence noetherian. Applying Lemma 2.1.17(2) to the R-bilinear form ${}_{\psi} < -, - >$, we get some $p \in \mathrm{Hom}_{\mathcal{C}}(L, M)$ such that ${}_{\varphi} < f_0, - >= {}_{\psi} < p, - >$. That is, for any $i \in \mathrm{Hom}_{\mathcal{A}}(Z[-1], L)$, we have

$${}_{\varphi}< f_0, i >= {}_{\psi} < p, i >= \psi(pi) = \varphi(fpi) = {}_{\varphi} < fp, i > .$$

Since ${}_{\varphi} < -, - >$ is non-degenerate, $f_0 = fp$. This shows that f is a right $\mathcal{C}$-approximation of X. Since M is strongly indecomposable, f is right minimal.

Finally, consider a morphism $h : L \to Y$ in $\mathcal{A}$ with $L \in \mathcal{C}$. Since v is not a retraction, neither is vh. Thus, there exists $i : L \to N$ such that $vh = ti = vgi$. That is, $v(h - gi) = 0$. By Proposition 1.1.5, there exists $j : L \to X$ such that $uj = h - gi$. Since we have just proved that f is a right $\mathcal{C}$-approximation of X, we can obtain $k : L \to M$ such that $j = fk$. Therefore, we have $h - gi = uj = ufk = grk$. Thus, $h = g(i + rk)$. That is, g is a right $\mathcal{C}$-approximation of Y. The proof of the theorem is completed.

We will say that $\mathcal{C}$ has *right Auslander-Reiten triangles* if each of its strongly indecomposable non-Ext-projective objects is the ending term of an Auslander-Reiten triangle in $\mathcal{C}$; and it has *left Auslander-Reiten triangles* if each of its strongly indecomposable non-Ext-injective objects is the starting term of an Auslander-Reiten triangle in $\mathcal{C}$. We know that an Auslander-Reiten triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{C}$ is unique up to isomorphism for X and for Z, we shall write $X = \tau_{\mathcal{A}} Z$ and $Z = \tau_{\mathcal{A}}^{-} X$. The uniqueness is well known; see, for example [15, (3.5)].

2.2.7 Theorem. *Let $\mathcal{A}$ be a triangulated R-category where R is complete local noetherian ring, and let $\mathcal{C}$ be an extension-closed subcategory of $\mathcal{A}$ which is Hom-finite and Krull-Schmidt.*

(1) *If $\mathcal{A}$ has right Auslander-Reiten triangles, then $\mathcal{C}$ has right Auslander-Reiten triangles if and only if $\tau_{\mathcal{A}} Z$ has a right minimal $\mathcal{C}$-approximation, for any indecomposable non-Ext-projective object Z in $\mathcal{C}$.*

(2) *If $\mathcal{A}$ has left Auslander-Reiten triangles, then $\mathcal{C}$ has left Auslander-Reiten triangles if and only if $\tau_{\mathcal{A}}^{-} X$ has a left minimal $\mathcal{C}$-approximation, for any indecomposable object non-Ext-injective X in $\mathcal{C}$.*

Proof. We shall only prove (1). Let $Z \in \mathcal{C}$ be indecomposable and non-Extprojective. Since $\mathcal{A}$ is Krull-Schmidt, Z is strongly indecomposable. We assume that $\mathcal{C}$ has right Auslander-Reiten triangles. Then there exists an Auslander-Reiten triangle $M \longrightarrow N \longrightarrow Z \longrightarrow M[1]$ in $\mathcal{C}$. For any $L \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(L, M)$ is of finite length. By Theorem 2.2.6, $\tau_{\mathcal{A}} Z$ has a minimal right $\mathcal{C}$-approximation.

Conversely, by assumption, there exists an Auslander-Reiten triangle $\tau_{\mathcal{A}}Z \longrightarrow Y \longrightarrow Z \longrightarrow \tau_{\mathcal{A}}Z[1]$ in $\mathcal{A}$ and a right minimal $\mathcal{C}$-approximation $f : M \to \tau_{\mathcal{A}}Z$ of $\tau_{\mathcal{A}}Z$. By Proposition 2.2.5, $\mathcal{C}$ has an Auslander-Reiten triangle ending with Z. The proof of the theorem is completed.

Chapter 3

Auslander-Reiten Triangles in Triangulated Categories with a t-structure

In this chapter, our main objective is to study Auslander-Reiten triangles in a triangulated category with a t-structure. One part of this chapter is to show an application of some results obtained in Chapter 2 on Auslander-Reiten triangles in subcategories, and the other one is to study the Auslander-Reiten triangles in a triangulated category which possesses a particular property called *t-hereditary*.

3.1 The t-structure

Now we recall some terminology and basic facts from [23].

Let $\mathcal{D}$ be a triangulated category with translation functor T. We say that a full subcategory of $\mathcal{D}$ is *strictly full* if it is closed under isomorphisms. If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a pair of strictly full subcategories of $\mathcal{D}$, then we put $\mathcal{D}^{\leq n} = T^{-n}(\mathcal{D}^{\leq 0})$ and $\mathcal{D}^{\geq n} = T^{-n}(\mathcal{D}^{\geq 0})$, for any $n \in \mathbb{Z}$.

3.1.1 DEFINITION. Let $\mathcal{D}$ be a triangulated category. A pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of $\mathcal{D}$ is called a *t-structure* if the following conditions are verified:

(1) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$.

(2) $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$, for any $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

(3) For any object X in $\mathcal{D}$, there exists an exact triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$

where $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

The following result is well known; see, for example, [1, (1.1)]

3.1.2 THEOREM. *Let $\mathcal{D}$ be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

(1) *For any $n \in \mathbb{Z}$, the subcategory $\mathcal{D}^{\leq n}$ is stable under T and the subcategory $\mathcal{D}^{\geq n}$ is stable under T^{-1}.*

(2) *The canonical inclusion $\mathcal{D}^{\leq n} \to \mathcal{D}$ has a right adjoint $\tau_{\leq n}$, and the canonical inclusion $\mathcal{D}^{\geq n} \to \mathcal{D}$ has a left adjoint $\tau_{\geq n}$ such that, for any $X \in \mathcal{D}$, we have $X \in \mathcal{D}^{\leq n}$ if and only if $\tau_{\geq n+1}(X) = 0$; and $X \in \mathcal{D}^{\geq n}$ if and only if $\tau_{\leq n-1}(X) = 0$.*

(3) *For any $n \in \mathbb{Z}$, the subcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n+1}$ are extension-closed subcategories such that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq n}$ and $Y \in \mathcal{D}^{\geq n+1}$.*

The following result is well know; see, for example, [23, Chapter 4, (1.2.1), (2.1.1)].

3.1.3 THEOREM. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$.*

(1) *$\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is an abelian category, called the* heart *of $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.*

(2) *Any short exact sequence* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *in* $\mathcal{H}$ *induces an exact triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$ *in* $\mathcal{D}$*, in such a way that we have an isomorphism of abelian groups*

$$\mathrm{Ext}^1_{\mathcal{H}}(Z, X) \cong \mathrm{Hom}_{\mathcal{D}}(Z, X[1]).$$

The following result is a reformulation of a well-known result; see, for example, [23, Chapter 4, (1.2.5)].

3.1.4 Proposition. *Let* $\mathcal{D}$ *be a triangulated category with a t-structure* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$*. For any* $X \in \mathcal{D}$ *and* $n \in \mathbb{Z}$*, there exists a canonical exact triangle*

$$\tau_{\leq n}(X) \xrightarrow{q} X \xrightarrow{p} \tau_{\geq n+1}(X) \longrightarrow \tau_{\leq n}(X)[1],$$

where q *is a minimal right* $\mathcal{D}^{\leq n}$*-approximation of* X*, and* p *is a minimal left* $\mathcal{D}^{\geq n+1}$*-approximation of* X*.*

Proof. The existence of the exact triangle is well-known. We shall prove only that $q : \tau_{\leq n}(X) \to X$ is a right minimal $\mathcal{D}^{\leq n}$-approximation of X. Let $f : Y \to X$ be a morphism in $\mathcal{D}$ with $Y \in \mathcal{D}^{\leq n}$. Then we obtain a commutative diagram as follows:

$$\begin{tikzcd}
Y \arrow[r, "1_Y"] \arrow[d, "\tau_{\leq n}(f)"] & Y \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
\tau_{\leq n}(X) \arrow[r, "q"] & X \arrow[r, "p"] & \tau_{\geq n+1}(X) \arrow[r, "\pi"] & \tau_{\leq n}(X)[1].
\end{tikzcd}$$

In particular, $f = q \circ \tau_{\leq n}(f)$. This shows that q is a right $\mathcal{D}^{\leq n}$–approximation of X. Let $g : \tau_{\leq n}(X) \to \tau_{\leq n}(X)$ such that $q = qg$. Then, $q(1 - g) = 0$. Therefore, $1 - g = \pi[-1]h$ for some $h : \tau_{\leq n}(X) \to \tau_{\geq n+1}(X)[-1]$. But, since $\tau_{\geq n+1}(X)[-1] \in \mathcal{D}^{\geq n+1}$, we get $h = 0$. That is $1 - g = 0$. The proof of the proposition is completed.

We shall need the following statement.

3.1.5 Lemma. *Let $\mathcal{D}$ be a triangulated category with an object X. If $\mathcal{D}$ has an exact triangle $Y \xrightarrow{u} X \xrightarrow{v} Z \xrightarrow{w} Y[1]$ with $Y \in \mathcal{D}^{\le n}$ and $Z \in \mathcal{D}^{\ge n+1}$, then there exists a commutative diagram*

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "f"] & X \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h"] & Y[1] \arrow[d, "f[1]"] \\
\tau_{\le n}(X) \arrow[r, "q"] & X \arrow[r, "p"] & \tau_{\ge n+1}(X) \arrow[r] & \tau_{\le n}(X)[1],
\end{tikzcd}$$

which is an isomorphism of triangles.

Proof. Let $Y \xrightarrow{u} X \xrightarrow{v} Z \xrightarrow{w} Y[1]$ be an exact triangle in $\mathcal{D}$ with $Y \in \mathcal{D}^{\le n}$ and $Z \in \mathcal{D}^{\ge n+1}$. By Theorem 3.1.2(3), $\mathrm{Hom}_{\mathcal{D}}(Y, \tau_{\ge n+1}(X)) = 0$. Therefore, $pu = 0$ and there exists a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "f"] & X \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h"] & Y[1] \arrow[d, "f[1]"] \\
\tau_{\le n}(X) \arrow[r, "q"] & X \arrow[r, "p"] & \tau_{\ge n+1}(X) \arrow[r] & \tau_{\le n}(X)[1].
\end{tikzcd}$$

On the other hand, since $\mathrm{Hom}_{\mathcal{D}}(\tau_{\le n}(X), Z) = 0$, we obtain another commutative diagram

$$\begin{tikzcd}
\tau_{\le n}(X) \arrow[r, "q"] \arrow[d, "f'"] & X \arrow[r, "p"] \arrow[d, equal] & \tau_{\ge n+1}(X) \arrow[r] \arrow[d, "h'"] & \tau_{\le n}(X)[1] \arrow[d, "f'[1]"] \\
Y \arrow[r, "u"] & X \arrow[r, "v"] & Z \arrow[r, "w"] & Y[1].
\end{tikzcd}$$

Composing these two diagrams, we obtain a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "f'f"] & X \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "h'h"] & Y[1] \arrow[d, "(f'f)[1]"] \\
Y \arrow[r, "u"] & X \arrow[r, "v"] & Z \arrow[r, "w"] & Y[1].
\end{tikzcd}$$

This yields $(1 - f'f)u = 0$, and hence $1 - f'f$ factors through $Z[-1]$. However, since $Z[-1] \in \mathcal{D}^{\ge n+1}$, we know that $\mathrm{Hom}_{\mathcal{D}}(Y, Z[-1]) = 0$. Therefore, $1 - f'f = 0$, that is, $f'f = 1$. Similarly, $ff' = 1$. This shows that f is an isomorphism, and hence, so is h. The proof of the lemma is completed.

We can obtain the following result by applying Proposition 2.2.5 in the previous chapter.

3.1.6 THEOREM. *Let $\mathcal{D}$ be a triangulated category with t-structure* $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, *and let $\mathcal{D}$ have an Auslander-Reiten triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

(1) *If $Z \in \mathcal{D}^{\leq n}$ is not Ext-projective for some $n \in \mathbb{Z}$, then $\mathcal{D}^{\leq n}$ has an Auslander-Reiten triangle*:

$$\tau_{\leq n}(X) \longrightarrow \tau_{\leq n}(Y) \longrightarrow Z \longrightarrow \tau_{\leq n}(X)[1].$$

(2) *If $X \in \mathcal{D}^{\geq n}$ is not Ext-injective for some $n \in \mathbb{Z}$, then $\mathcal{D}^{\geq n}$ has an Auslander-Reiten triangle*:

$$X \longrightarrow \tau_{\geq n}(Y) \longrightarrow \tau_{\geq n}(Z) \longrightarrow X[1].$$

Proof. We shall only prove Statement (1). Suppose that $Z \in \mathcal{D}^{\leq n}$ is not Ext-projective. Consider the exact triangle

$$(*) \qquad \tau_{\leq n}(X) \xrightarrow{q} X \xrightarrow{p} \tau_{\geq n+1}(X) \longrightarrow \tau_{\leq n}(X)[1].$$

By the adjunction stated in Theorem 3.1.2(2), there exists an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq n}(X), X) \to \mathrm{End}_{\mathcal{D}}(\tau_{\leq n}(X))$$

On the other hand, since $Z \in \mathcal{D}^{\leq n}$, $\mathrm{Hom}_{\mathcal{D}}(Z, \tau_{\geq n+1}(X)) = 0$. By Lemma 2.1.11(3), $\mathrm{Hom}_{\mathcal{D}}(\tau_{\geq n+1}(X), X[1]) = 0$. Thus, applying $\mathrm{Hom}_{\mathcal{D}}(-, X)$ to $(*)$ gives rise to an epimorphism

$$\mathrm{Hom}_{\mathcal{D}}(q, X) : \mathrm{End}_{\mathcal{D}}(X) \to \mathrm{Hom}_{\mathcal{D}}(\tau_{\leq n}(X), X).$$

Composing this with φ, we get an epimorphism

$$\phi : \mathrm{End}_{\mathcal{D}}(X) \to \mathrm{End}_{\mathcal{D}}(\tau_{\leq n}(X)),$$

which is a ring morphism. Since Z is not Ext-projective, by Proposition 2.2.5(1), $\tau_{\leq n}(X) \neq 0$. Since $\mathrm{End}_{\mathcal{D}}(X)$ is local, $\mathrm{End}_{\mathcal{D}}(\tau_{\leq n}(X))$ is also local.

Thus, $\tau_{\leq n}(X)$ is strongly indecomposable. By Proposition 2.2.5, we have a commutative diagram

$$\begin{tikzcd}
\tau_{\leq n}(X) \arrow[r, "u'"] \arrow[d, "q"] & N \arrow[r, "v'"] \arrow[d, "g"] & Z \arrow[r, "w'"] \arrow[d, equal] & \tau_{\leq n}(X)[1] \arrow[d, "q[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1]
\end{tikzcd}$$

in $\mathcal{D}$, where the upper row is an Auslander-Reiten triangle in $\mathcal{D}^{\leq n}$. Considering $w = w'q[1]$, by the octahedral axiom, we get a commutative diagram

$$\begin{tikzcd}
Z[-1] \arrow[r, "-w'[-1]"] \arrow[d, equal] & \tau_{\leq n}(X) \arrow[r, "u'"] \arrow[d, "q"] & N \arrow[r, "v'"] \arrow[d] & Z \arrow[d, equal] \\
Z[-1] \arrow[r, "-w[-1]"] \arrow[d, "-w'[-1]"] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d] & Z \arrow[d] \\
\tau_{\leq n}(X) \arrow[r, "q"] \arrow[d, "u'"] & X \arrow[r, "p"] \arrow[d] & \tau_{\geq n+1}(X) \arrow[r] \arrow[d, equal] & \tau_{\leq n}(X)[1] \arrow[d] \\
N \arrow[r] & Y \arrow[r] & \tau_{\geq n+1}(X) \arrow[r] & N[1],
\end{tikzcd}$$

where all the rows are exact triangles. Since $N \in \mathcal{D}^{\leq n}$ and $\tau_{\geq n+1}(X) \in \mathcal{D}^{\geq n+1}$, by Lemma 3.1.5, we get $N \cong \tau_{\leq n}(Y)$. Thus, there exists an Auslander-Reiten triangle

$$\tau_{\leq n}(X) \longrightarrow \tau_{\leq n}(Y) \longrightarrow Z \longrightarrow \tau_{\leq n}(X)[1].$$

The proof of the theorem is completed.

3.2 t-structure with a t-hereditary Heart

In this section, our main objective is to study the Auslander-Reiten triangles in a triangulated category which possesses a property called t-heredity.

Throughout this section, let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ be a t-structure and $\mathcal{H}$ be its heart. First we define an additive functor $H^0 : \mathcal{D} \to \mathcal{H}$ by

$$H^0(X) = \tau_{\leq 0}(\tau_{\geq 0}(X));$$

and, for any $p \in \mathbb{Z}$, we define an additive functor $H^p : \mathcal{D} \to \mathcal{H}$ by

$$H^p(X) = H^0(X[p]).$$

3.2.1 Definition. Let $\mathcal{D}$ be a triangulated category. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is called *bounded* if it satisfies the following conditions.

(1) For any $X \in \mathcal{D}$, if $H^p(X) = 0$ for all $p \in \mathbb{Z}$, then $X = 0$.

(2) $H^p(X) = 0$, for all but finitely many $p \in \mathbb{Z}$.

3.2.2 Definition. Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure with heart $\mathcal{H}$. We say that $\mathcal{H}$ is *t-hereditary* if $\mathrm{Hom}_{\mathcal{D}}(M, N[n]) = 0$, for all $M, N \in \mathcal{H}$ and $n \geq 2$.

The following result is well known; see, for example, [23, Chapter 4, (2.3.1)].

3.2.3 Theorem. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with a t-hereditary heart $\mathcal{H}$. For any object X in $\mathcal{D}$,*

$$X \cong \bigoplus_{p\in\mathbb{Z}} H^p(X)[-p].$$

By this theorem, we obtain the following lemma.

3.2.4 Lemma. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with a t-hereditary heart $\mathcal{H}$. Then* $\mathrm{Hom}_{\mathcal{D}}(X, Y[p]) = 0$, *for any $X, Y \in \mathcal{H}$ and $p \in \mathbb{Z}\backslash\{0, 1\}$.*

Proof. Let X, Y be two objects in $\mathcal{H}$. If $p \geq 2$, since $\mathcal{H}$ is t-hereditary, $\mathrm{Hom}_{\mathcal{D}}(X, Y[p]) = 0$. Now consider $p \leq -1$. That is, $-p \geq 1$. We know that $Y[p] \in \mathcal{D}^{\leq -p} \cap \mathcal{D}^{\geq -p}$. In particular, $X \in \mathcal{D}^{\leq 0}$ and $Y[p] \in \mathcal{D}^{\geq 1}$. Therefore, $\mathrm{Hom}_{\mathcal{D}}(X, Y[p]) = 0$. The proof of the lemma is completed.

3.2.5 COROLLARY. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with a t-hereditary heart $\mathcal{H}$. Then*

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(H^0(X) \oplus H^1(X)[-1], Y),$$

for any $X \in \mathcal{D}$ and $Y \in \mathcal{H}$.

Proof. Suppose that $X \in \mathcal{D}$ and $Y \in \mathcal{H}$. By Theorem 3.2.3, we know that

$$X \cong \bigoplus_{p \in \mathbb{Z}} H^p(X)[-p].$$

Then by Lemma 3.2.4, $\mathrm{Hom}_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(H^0(X) \oplus H^1(X)[-1], Y)$. The proof of the lemma is completed.

The following result tells us how the minimal almost split morphisms in $\mathcal{D}$ are related to those in $\mathcal{H}$, which is inspired from [9, (7.1)].

3.2.6 LEMMA. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with a t-hereditary heart $\mathcal{H}$. Let $X, Y, Z \in \mathcal{H}$.*

(1) *If $(f, \eta)^T : X \to Y \oplus Z[1]$ is a minimal left almost split morphism in $\mathcal{D}$, then $f : X \to Y$ is a left almost split morphism in $\mathcal{H}$.*

(2) *If $(g, \varsigma) : Y \oplus Z[-1] \to X$ is a minimal right almost split morphism in $\mathcal{D}$, then $g : Y \to X$ is a minimal right almost split morphism in $\mathcal{H}$.*

Proof. Let $(f, \eta)^T : X \to Y \oplus Z[1]$ be a minimal left almost split morphism in $\mathcal{D}$. It is evident that $f : X \to Y$ is left minimal and is not a section. Suppose that $u : X \to M$ is a non-section morphism in $\mathcal{H}$. Then u factor through $(f, \eta)^T$ in $\mathcal{D}$. By Lemma 3.2.4, $\mathrm{Hom}_{\mathcal{D}}(Z[1], M) = 0$. We see that u factor through f in $\mathcal{H}$. This proves Statement (1). In a dual manner, we can establish Statement (2). The proof of the lemma is completed.

For later use, we quote the following result from [9, (2.1)].

3.2.7 Lemma. *Let $\mathcal{H}$ be an abelian category with a short exact sequence*

$$0 \longrightarrow X \xrightarrow{q} Y \xrightarrow{p} Z \longrightarrow 0 .$$

(1) *The morphism q is minimal right almost split if and only if Z is simple and p is its projective cover. In this case, we write $X = \mathrm{rad}P$.*

(2) *The morphism p is minimal left almost split if and only if X is simple and q is its injective envelope.*

Now we are ready to obtain our main result of this section, which is analogous to a result by Bautista, Liu and Paquette for the bounded derived category of an abelian category with a similar argument; see [9, (7.4)].

3.2.8 Theorem. *Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with a t-hereditary heart $\mathcal{H}$.*

(1) *If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an Auslander-Reiten sequence in $\mathcal{H}$, then $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ is an Auslander-Reiten triangle in $\mathcal{D}$.*

(2) *If S is a simple object in $\mathcal{H}$ having a projective cover P and injective hull I in $\mathcal{H}$, then $\mathcal{D}$ has an Auslander-Reiten triangle as follows:*

$$I[-1] \longrightarrow (I/S)[-1] \oplus \mathrm{rad}P \longrightarrow P \longrightarrow I.$$

(3) *Every Auslander-Reiten triangle in $\mathcal{D}$ is a shift of an Auslander-Reiten triangle as stated in the above two statements.*

Proof. (1) Let $\eta : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be an Auslander-Reiten sequence in $\mathcal{H}$. Then, by Theorem 3.1.3(2),

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\eta} X[1]$$

is an exact triangle in $\mathcal{D}$. Let U be an object in $\mathcal{D}$. By Corollary 3.2.5, we get

$$\mathrm{Hom}_{\mathcal{D}}(U, Y) \cong \mathrm{Hom}_{\mathcal{D}}(H^0(U) \oplus H^1(U)[-1], Y).$$

It is enough to show that for any non-zero non-retraction morphism

$$u = (u_0, u_1) : H^0(U) \oplus H^1(U)[-1] \to Z \in \mathcal{D},$$

u factors through g. Since $\mathcal{D}$ is t-hereditary,

$$\mathrm{Hom}(H^1(U)[-1], X[1]) \cong \mathrm{Hom}(H^1(U), X[2]) = 0.$$

Thus, u_1 factors through g. Now consider $u_0 : H^0(U) \to Z$. It is obvious that u_0 is not a retraction. Since g is right almost split in $\mathcal{H}$, there exists a morphism $v : H^0(U) \to Y$ such that $u_0 = gv$. Therefore,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

is an Auslander-Reiten triangle in $\mathcal{D}$.

(2) Let S be a simple object in $\mathcal{H}$ with projective cover $\varepsilon : P \to S$ and injective envelope $\iota : S \to I$. In view of Lemma 3.2.7, we deduce that P and I are strongly indecomposable. Setting $h = \iota\varepsilon$, we get an exact triangle

$$I[-1] \xrightarrow{f} M \xrightarrow{g} P \xrightarrow{h} I$$

in $\mathcal{D}$. Let $\mu : X \to P$ be a non-zero non-retraction morphism in $\mathcal{D}$. By Corollary 3.2.5, we may assume that $u = (u_0, u_1) : H^0(X) \oplus H^1(X)[-1] \to P$. Then u_0 is a non-retraction in $\mathcal{H}$. Thus $\varepsilon u_0 = 0$, and hence $hu_0 = 0$. On the other hand, by Theorem 3.1.3(3),

$$\mathrm{Hom}_{\mathcal{D}}(H^1(X)[-1], I) \cong \mathrm{Hom}_{\mathcal{D}}(H^1(X), I[1]) \cong \mathrm{Ext}^1_{\mathcal{H}}(H^1(X), I) = 0.$$

In particular, $hu_1 = 0$, and consequently, $hu = 0$. Therefore, g is right almost split in $\mathcal{D}$. Since $I[-1]$ is strongly indecomposable,

$$I[-1] \xrightarrow{f} M \xrightarrow{g} P \xrightarrow{h} I \ \in \mathcal{D}$$

is an Auslander-Reiten triangle in $\mathcal{D}$. Again, by Corollary 3.2.5, we may assume that $M = H^0(M) \oplus H^1(M)[-1]$. Write

$$f = (f_1, f_2[-1])^T \text{ and } g = (g_1, g_2),$$

where $f_2 : I \to H^1(M)$ and $g_1 : H^0(X) \to P$ are morphism in $\mathcal{H}$. By Lemma 3.2.6, f_2 is minimal right almost split, and g_1 is minimal right almost split. By Lemma 3.2.7, we get $H^0(X) \cong \mathrm{rad}P$ and $H^1(X) \cong I/S$.

(3) Let $\delta :\ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be an Auslander-Reiten triangle in $\mathcal{D}$. In particular, X, Z are strongly indecomposable. In view of Theorem 3.2.3, up to a shift, we may assume that $X \in \mathcal{H}$. Since f is left minimal, we may assume that $Y = M \oplus N[1]$ with $M, N \in \mathcal{H}$. Write $f = (u, \zeta)^T : X \to M \oplus N[1]$ with $u : X \to M$ a morphism in $\mathcal{H}$, and $g = (\xi, v) : M \oplus N[1] \to Z$. By Lemma 3.2.6, u is minimal left almost split in $\mathcal{H}$.

First, we suppose that X is not injective in $\mathcal{H}$. Then there exists a non-section monomorphism $X \to L$ in $\mathcal{H}$, which factors through u. Thus, u is a monomorphism. Hence, $\mathcal{H}$ has a short exact sequence

$$0 \longrightarrow X \longrightarrow M \longrightarrow N \longrightarrow 0$$

which is an Auslander-Reiten sequence by Lemma 2.0.3. By statement (1), $X \longrightarrow M \longrightarrow N \longrightarrow X[1]$ is an Auslander-Reiten triangle in $\mathcal{D}$, which is isomorphic to the Auslander-Reiten triangle δ. That is, δ is the form of statement (1).

Next, we suppose that $X = I$, an injective object in $\mathcal{H}$. Then $u : I \to M$ is a minimal left almost split epimorphism in $\mathcal{H}$. Let $q : S \to I$ be the kernel of u. By Lemma 3.2.7, S is simple with q being its injective envelope and $M \cong I/S$. Suppose that $Z \in \mathcal{H}$. Then $v = 0$ and $\xi u = -v\zeta = 0$. Since u is an epimorphism in $\mathcal{H}$, we get $\xi = 0$, and hence $g = (\xi, v) = 0$. As a consequence, $h : Z \to I[1]$ is a section, which contradicts $\mathrm{Hom}_{\mathcal{D}}(Z, I[1]) \cong \mathrm{Ext}^1_{\mathcal{H}}(I, Z) = 0$. Therefore, Z is not in $\mathcal{H}$. Since $h \neq 0$ and Z is strongly indecomposable, $Z = P[1]$ for some $P \in \mathcal{H}$. Then, $h = s[1]$ and $v = j[1]$, where $s : P \to I$ and $j : N \to P$ are morphisms in $\mathcal{H}$. Now,

$$\delta[-1] : \quad I[-1] \xrightarrow{f[-1]} Y[-1] \xrightarrow{g[-1]} P \xrightarrow{s} I,$$

where $g[-1] = (\xi[-1], j) : M[-1] \oplus N \to P$, is an Auslander-Reiten triangle in $\mathcal{D}$. By Lemma 3.2.7, $j : N \to P$ is minimal right almost split in $\mathcal{H}$. If P is not projective, as argued in the case where X is not injective, we can show that $\delta[-1]$ is isomorphic to an Auslander-Reiten triangle induced from an Auslander-Reiten sequence in $\mathcal{H}$ ending with P. In particular, $I[-1]$ is isomorphic to an object in $\mathcal{H}$, which is a contradiction. Thus, P is projective. Therefore, $j : N \to P$ is a minimal right almost split monomorphism. Let $\varepsilon : P \to T$ be the cokernel of j. By Lemma 3.2.7, T is simple with ε being its projective cover, and $N \cong \mathrm{rad}P$. Moreover, since $s \circ g[-1] = 0$ and $fs = 0$, we have $sj = 0$ and $us = 0$. This yields a factorization $s = qp\varepsilon$, where $p : T \to S$ is a non-zero morphism in $\mathcal{H}$. Since T, S are simple, p is an isomorphism. That is, δ is of the form stated in Statement (2). The proof of the theorem is completed.

Chapter 4

Auslander-Reiten Theory in Derived Categories

In this chapter, our main objective is to generalize a well-known result of Happel on the existence of Auslander-Reiten triangles in the bounded derived category of finite dimensional modules over a finite dimensional algebra. We shall show that a similar result holds in the bounded derived category of all modules of a noetherian algebra over a complete local noetherian commutative ring.

4.1 Derived Categories

In this section, we introduce the derived category of an additive full subcategory of an abelian category and collect some basic facts. Throughout this section, $\mathcal{A}$ stands for an additive full subcategory of an abelian category $\mathscr{A}$.

4.1.1 Definition. A *complex* $(X^\bullet, d_X^\bullet)$, or simply $X^\bullet$, over $\mathcal{A}$ is a double infinite chain

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots, \ n \in \mathbb{Z}$$

of morphisms in $\mathcal{A}$ such that $d_X^{n+1}d_X^n = 0$, for all n, where X^n is an object called the *component of degree* n *of* $X^\bullet$, and d_X^n is a morphism called the *differential of degree* n.

4.1.2 Definition. A complex $X^\bullet$ over $\mathcal{A}$ is called *bounded-above* if $X^n = 0$ for all but finitely many positive integers n, and *bounded* if $X^n = 0$ for all but finitely many integers n.

4.1.3 Definition. Let $X^\bullet$ be a complex over $\mathcal{A}$. For each $n \in \mathbb{Z}$, the *n-th cohomology* of $X^\bullet$ is defined to be

$$H^n(X^\bullet) = \mathrm{Ker}(d_X^n)/\mathrm{Im}(d_X^{n-1}) \in \mathscr{A}.$$

One says that $X^\bullet$ has *bounded cohomology* if $H^n(X^\bullet) = 0$ for all but finitely many integers and that $X^\bullet$ is *acyclic* if $H^n(X^\bullet) = 0$ for all integers n.

4.1.4 Definition. A *morphism* $f^\bullet : X^\bullet \to Y^\bullet$ of complexes over $\mathcal{A}$ consists of morphisms $f^n : X^n \to Y^n, n \in \mathbb{Z}$, such that $d_Y^n f^n = f^{n+1} d_X^n$ for all $n \in \mathbb{Z}$.

The complexes over $\mathcal{A}$ form an additive category $C(\mathcal{A})$. For $X^\bullet \in C(\mathcal{A})$ and $s \in \mathbb{Z}$, the *shift* of $X^\bullet$ by s, denoted by $X^\bullet[s]$, is the complex of which the component of degree n is X^{n+s} and the differential of degree n is $(-1)^s d_X^{n+s}$.

4.1.5 Definition. The automorphism of $C(\mathcal{A})$ sending $X^\bullet$ to $X^\bullet[1]$ is called the *shift functor* of $C(\mathcal{A})$.

4.1.6 Definition. A morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C(\mathcal{A})$ is called a *quasi-isomorphism* if f^n induces an isomorphism $H^n(f) : H^n(X^\bullet) \to H^n(Y^\bullet)$ for every $n \in \mathbb{Z}$; and *null-homotopic* if there exist $h^n : X^n \to Y^{n-1}, n \in \mathbb{Z}$, such that $f^n = h^{n+1} d_X^n + d_Y^{n-1} h^n$, for all $n \in \mathbb{Z}$.

4.1.7 Lemma. *Let $f^\bullet : X^\bullet \to Y^\bullet$, $g^\bullet : Y^\bullet \to Z^\bullet$ and $h^\bullet : Z^\bullet \to U^\bullet$ be morphisms in $C(\mathcal{A})$. If $g^\bullet f^\bullet$ and $h^\bullet g^\bullet$ are quasi-isomorphisms, then $g^\bullet$ is a quasi-isomorphism.*

Proof. Suppose that $g^\cdot f^\cdot$ and $h^\cdot g^\cdot$ are quasi-isomorphisms. It follows that $H^n(g^\cdot f^\cdot)$ and $H^n(h^\cdot g^\cdot)$ are isomorphisms, for all $n \in \mathbb{Z}$. Therefore, there exist morphisms α_n and β_n such that $1_{H^n(Z^\bullet)} = H^n(g^\cdot f^\cdot)\alpha_n = H^n(g^\cdot)H^n(f^\cdot)\alpha_n$ and $1_{H^n(Y^\bullet)} = \beta_n H^n(h^\cdot g^\cdot) = \beta_n H^n(h^\cdot)H^n(g^\cdot)$, for all $n \in \mathbb{Z}$. Hence, $H^n(g^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$. That is, $g^\cdot$ is a quasi-isomorphism. The proof of the lemma is completed.

4.1.8 Lemma. *Let $f^\cdot : X^\bullet \to Y^\bullet$ and $g^\cdot : Y^\bullet \to Z^\bullet$ be two morphisms in $C(\mathcal{A})$. If $h^\cdot = g^\cdot f^\cdot$ is a quasi-isomorphism, then $f^\cdot$ is a quasi-isomorphism if and only if $g^\cdot$ is a quasi-isomorphism.*

Proof. Suppose that $h^\cdot = g^\cdot f^\cdot$ is a quasi-isomorphism in $C(\mathcal{A})$. Then $H^n(h^\cdot) = H^n(g^\cdot)H^n(f^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$. If $H^n(f^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$, then $H^n(g^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$. Conversely, if $H^n(g^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$, then $H^n(f^\cdot)$ is an isomorphism for all $n \in \mathbb{Z}$. The proof of the lemma is completed.

The following result is well known; see, for example, [23, Chapter 3, (1.3.1), (1.3.2)].

4.1.9 Lemma. *The class of null-morphisms forms an ideal in $C(\mathcal{A})$.*

The full subcategory of $C(\mathcal{A})$ generated by the bounded-above complexes and by the bounded complexes will be denoted by $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$, respectively. Moreover, $C^{-,b}(\mathcal{A})$ denotes the full subcategory of $C^-(\mathcal{A})$ generated by the complexes of bounded cohomology.

From now on, we fix $* \in \{\emptyset, -, b, \{-, b\}\}$.

4.1.10 Definition. Let $f^\cdot : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The *mapping cone* of $f^\cdot$, denoted by $C^\bullet_f$, is the complex of which the component of degree n is given by $Y^n \oplus X^{n+1}$ and the differential of degree n is given by

$$d^n_{C_f} = \begin{pmatrix} d^n_Y & f^{n+1} \\ 0 & -d^{n+1}_X \end{pmatrix}, \qquad n \in \mathbb{Z}.$$

The mapping cone of $f^\bullet$ fits into a short exact sequence

$$0 \longrightarrow Y^\bullet \xrightarrow{i_f^\bullet} C_f^\bullet \xrightarrow{p_f^\bullet} X^\bullet[1] \longrightarrow 0$$

in $C^*(\mathcal{A})$, where $i_f^\bullet : Y^\bullet \to C_f^\bullet$ is the canonical injection, given by

$$i_f^n = \begin{pmatrix} 1_{Y^n} \\ 0 \end{pmatrix} : Y^n \to Y^n \oplus X^{n+1}, \qquad n \in \mathbb{Z};$$

and $p_f^\bullet : C_f^\bullet \to X^\bullet[1]$ is the canonical projection given by

$$p_f^n = (\,0 \ \ 1_{X^{n+1}}\,) : Y^n \oplus X^{n+1} \to X^{n+1}, \qquad n \in \mathbb{Z}.$$

4.1.11 Definition. The *homotopy category* $K^*(\mathcal{A})$ is the quotient category of $C^*(\mathcal{A})$ modulo the ideal of null-homotopic morphisms.

The canonical projection functor $P^*_{\mathcal{A}} : C^*(\mathcal{A}) \to K^*(\mathcal{A})$ is an additive functor. For a morphism $u^\bullet \in C^*(\mathcal{A})$, we write $\bar{u}^\bullet = P^*_{\mathcal{A}}(u^\bullet) \in K^*(\mathcal{A})$. Given a morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $C^*(\mathcal{A})$, the diagram

$$X^\bullet \xrightarrow{\bar{f}^\bullet} Y^\bullet \xrightarrow{\bar{i}_f^\bullet} C_f^\bullet \xrightarrow{\bar{p}_f^\bullet} X^\bullet[1]$$

is called a *standard triangle* in $K^*(\mathcal{A})$.

The following result is well-known; see, for example, [23, Chapter 3, (2.1.1)].

4.1.12 Theorem. *The homotopy category $K^*(\mathcal{A})$ is a triangulated category whose translation functor is the shift functor and whose exact triangles are the diagrams which are isomorphic to standard triangles.*

A *quasi-isomorphism* in $K^*(\mathcal{A})$ is the image of a quasi-isomorphism in $C^*(\mathcal{A})$ under $P^*_{\mathcal{A}}$. It is well known that the quasi-isomorphisms in $K^*(\mathcal{A})$ form a localizing class compatible with the triangulation, see, for example, Chapter 3, Proposition 3.1.2, [23].

4.1.13 Definition. The *derived category* $D^*(\mathcal{A})$ of $\mathcal{A}$ is the localization of $K^*(\mathcal{A})$ with respect to the quasi-isomorphisms.

Recall that the morphisms in $D^*(\mathcal{A})$ are the equivalence classes $\bar{f}^\bullet/\bar{s}^\bullet : X^\bullet \to Y^\bullet$ of the diagrams of morphisms

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "\bar{s}^\bullet"'] \arrow[dr, "\bar{f}^\bullet"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

in $K^*(\mathcal{A})$ with $s^\bullet$ a quasi-isomorphism. We have a canonical functor

$$L^*_{\mathcal{A}} : K^*(\mathcal{A}) \to D^*(\mathcal{A}),$$

call the *localization functor,* which sends $\bar{f}^\bullet$ to $\tilde{f}^\bullet = \bar{f}^\bullet/\bar{1}$.

4.1.14 Lemma. *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{A})$. Then $\tilde{f}^\bullet$ is an isomorphism in $D(\mathcal{A})$ if and only if $f^\bullet$ is a quasi-isomorphism.*

Proof. We only need to show the sufficiency. Assume that $\tilde{f}^\bullet$ is an isomorphism. Then there exists a morphism $\bar{g}^\bullet/\bar{t}^\bullet : Y^\bullet \to X^\bullet$ such that $(\bar{g}^\bullet/\bar{t}^\bullet)\tilde{f}^\bullet = \tilde{1}_{X^\bullet}$. In particular, there exits a morphism $\bar{h}^\bullet$ and a quasi-isomorphism $s^\bullet$ such that $(\bar{g}^\bullet/\bar{t}^\bullet)\tilde{f}^\bullet = \bar{g}^\bullet\bar{h}^\bullet/\bar{s}^\bullet = \tilde{1}_{X^\bullet}$. In view of the second equation, we have a commutative diagram

$$\begin{tikzcd} & M^\bullet \arrow[dl, "\bar{s}^\bullet"'] \arrow[dr, "\bar{g}^\bullet\bar{h}^\bullet"] & \\ X^\bullet & N^\bullet \arrow[u, "\bar{u}^\bullet"] \arrow[d, "\bar{s}^\bullet\bar{u}^\bullet"] & Y^\bullet \\ & X^\bullet \arrow[ul, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

such that $\bar{s}^\bullet\bar{u}^\bullet$ is a quasi-isomorphism. Since $\bar{s}^\bullet$ is a quasi-isomorphism, by Lemma 4.1.8, so is $\bar{u}^\bullet$. Therefore, $\bar{g}^\bullet(\bar{h}^\bullet\bar{u}^\bullet)$ is a quasi-isomorphism. On the other hand, since $\tilde{f}^\bullet(\bar{g}^\bullet/\bar{t}^\bullet) = \bar{f}^\bullet\bar{g}^\bullet/\bar{t}^\bullet = \tilde{1}_{Y^\bullet}$, using a similar argument, we obtain a quasi-isomorphism $\bar{v}^\bullet$ such that $\bar{f}^\bullet\bar{g}^\bullet\bar{v}^\bullet = \bar{t}^\bullet\bar{v}^\bullet$. By Lemma 4.1.8, we know that $f^\bullet g^\bullet$ and $g^\bullet h^\bullet$ are quasi-isomorphisms. Then, by Lemma 4.1.7,

$g^{\cdot}$ is a quasi-isomorphism. Using Lemma 4.1.8 again, we see that $f^{\cdot}$ is a quasi-isomorphism. The proof of the lemma is completed.

4.1.15 Corollary. *Let $\theta^{\cdot} = \bar{f}^{\cdot}/\bar{s}^{\cdot} : X^{\bullet} \to Y^{\cdot}$ be a morphism in $D(\mathcal{A})$. Then $\theta^{\cdot}$ is an isomorphism in and only if f is a quasi-isomorphism, and in this case, $H^n(X^{\bullet}) \cong H^n(Y^{\cdot})$ for all $n \in \mathbb{Z}$.*

Proof. By definition, $\theta^{\bullet} = \tilde{f}^{\cdot}(\tilde{s}^{\cdot})^{-1}$ and $\tilde{f}^{\bullet} = \theta^{\cdot}\tilde{s}^{\cdot}$. If f is a quasi-isomorphism, then $\tilde{f}$ is an isomorphism in $D(\mathcal{A})$, and so is $\theta^{\bullet}$. Conversely, if $\theta^{\cdot}$ is an isomorphism, then so is $\tilde{f}$. By Lemma 4.1.14, f is a quasi-isomorphism. Finally, write $s^{\cdot} : X^{\bullet} \to Z^{\bullet}$ and $f^{\cdot} : Z^{\bullet} \to Y^{\bullet}$. If $s^{\cdot}$ and $f^{\cdot}$ are quasi-isomorphisms, then $H^n(X^{\bullet}) \cong H^n(Z^{\cdot}) \cong H^n(Y^{\cdot})$, for all $n \in \mathbb{Z}$. The proof of the corollary is completed.

The following result is well known; see, for example, [23, Chapter 2, (1.6.1), Chapter 3, (3.2)].

4.1.16 Theorem. *The derived category $D^*(\mathcal{A})$ is a triangulated category, whose translation functor is the shift functor and whose exact triangles are the diagrams which are isomorphic to the images of the exact triangles in $K^*(\mathcal{A})$ under the localization functor $L^*_{\mathcal{A}}$.*

We denote by $E_{\mathcal{A}} : \mathcal{A} \to D^*(\mathcal{A})$ the canonical functor $E_{\mathcal{A}} = L^*_{\mathcal{A}} \circ P^*_{\mathcal{A}}$ which sends an object X to the stalk complex of which the component of degree zero is X and other components are zero, and sends a morphism $f : X \to Y$ to $\tilde{f} : X \to Y$. It is well known that $E_{\mathcal{A}} : \mathcal{A} \to D^*(\mathcal{A})$ is fully faithful; see, for example, [23, Chpter 3, (3.4.7)].

Suppose that $\mathcal{A}$ has a short exact sequence

$$\delta : \ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

It is easy to verify that the cone of the morphism $f : X \to Y$ in $C^*(\mathcal{A})$ is the complex

$$C^{\bullet}_f : \quad \cdots \longrightarrow 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow \cdots,$$

where Y is of degree 0. Let $p^\bullet : C_f^\bullet \to X[1]$ be the canonical projection as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots . \\
\cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, we have a quasi-isomorphism $s^\bullet : C_f^\bullet \to Z$ as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \cdots . \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

This yields a morphism $\delta : Z \to X[1]$ in $D^*(\mathcal{A})$ represented by the following diagram

$$\begin{tikzcd}
 & C_f^\bullet \arrow[dl, "\bar{s}^\bullet"'] \arrow[dr, "\bar{p}^\bullet"] & \\
Z & & X[1].
\end{tikzcd}$$

4.1.17 Proposition. *Let $\mathcal{A}$ have a short exact sequence*

$$\delta : \ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

Then it determines an exact triangle in $D^*(\mathcal{A})$ as follows:

$$X \xrightarrow{\tilde{f}} Y \xrightarrow{\tilde{g}} Z \xrightarrow{\delta} X[1].$$

Proof. By definition, we have an exact triangle

$$X \xrightarrow{\tilde{f}} Y \xrightarrow{\tilde{i}^\bullet} C_f^\bullet \xrightarrow{\tilde{p}^\bullet} X[1]$$

in $D^*(\mathcal{A})$, where $i^\bullet : Y \to C_f^\bullet$ is the canonical injection as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "1_Y"] & 0 \arrow[r] \arrow[d] & \cdots . \\
\cdots \arrow[r] & 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

By the definition of δ, we have $\delta \circ \tilde{s}^\bullet = \tilde{p}^\bullet$ in $D^*(\mathcal{A})$. This yields a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}"] \arrow[d, equal] & Y \arrow[r, "\tilde{i}^\bullet"] \arrow[d, equal] & C_f^\bullet \arrow[r, "\tilde{p}^\bullet"] \arrow[d, "\tilde{s}^\bullet"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\tilde{f}"] & Y \arrow[r, "\tilde{g}"] & Z \arrow[r, "\delta"] & X[1]
\end{tikzcd}$$

in $D^*(\mathcal{A})$. Since $\tilde{s}^\bullet$ is an isomorphism in $D^*(\mathcal{A})$, the lower row is an exact triangle in $D^*(\mathcal{A})$. The proof of the proposition is completed.

Remark. Although the above result is well known, the explicit description of the morphism δ seems to be new.

Now we introduce truncations of complexes. Consider a complex

$$X^\bullet : \quad \cdots \longrightarrow X^{p-1} \xrightarrow{d^{p-1}} X^p \xrightarrow{d^p} X^{p+1} \longrightarrow \cdots$$

in $C^*(\mathcal{A})$. For each $n \in \mathbb{Z}$, we define two kinds of truncation of complexes as follows:

$$\delta_{\geq n}(X^\bullet) : \quad \cdots \longrightarrow 0 \longrightarrow X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \longrightarrow \cdots$$

and

$$\delta_{<n}(X^\bullet) : \quad \cdots \longrightarrow X^{n-3} \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \longrightarrow 0 \longrightarrow \cdots .$$

Immediately, we obtain a morphism of complexes

$$\mu_n^\bullet : \delta_{\geq n}(X^\bullet) \to X^\bullet$$

such that $\mu_n^p = 1_{X^p}$, for $p \geq n$ and $\mu_n^p = 0$, for $p \leq n-1$; and a morphism of complexes

$$\pi_n^\bullet : X^\bullet \to \delta_{<n}(X^\bullet)$$

such that $\pi_n^p = 1_{X^p}$, for $p \leq n-1$ and $\pi_n^p = 0$, for $p \geq n$.

Moreover, we see that the mapping cone $C^\bullet_{\mu_n}$ of μ_n is the following complex:

$$\cdots \longrightarrow X^{n-3} \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{\binom{d^{n-2}}{0}} X^{n-1} \oplus X^n$$

$$\xrightarrow{d^{n-1}_{C_n}} X^n \oplus X^{n+1} \xrightarrow{d^n_{C_n}} X^{n+1} \oplus X^{n+2} \longrightarrow \cdots ,$$

where $d^p_{C_{\mu_n}}$ with $p \geq n-1$ is given by

$$d^p_{C_{\mu_n}} = \begin{pmatrix} d^p & 1_{X^{p+1}} \\ 0 & -d^{p+1} \end{pmatrix}.$$

The following result is stated in [14, (1.3)] without a proof. Hence, we include a proof here.

4.1.18 THEOREM. *Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathscr{A}$. For any complex $X^\bullet$ in $C^*(\mathcal{A})$ and $n \in \mathbb{Z}$, there exists an exact triangle*

$$\delta_{\geq n}(X^\bullet) \xrightarrow{\bar{\mu}_n^\bullet} X^\bullet \xrightarrow{\bar{\pi}_n^\bullet} \delta_{<n}(X^\bullet) \xrightarrow{\bar{\omega}_n^\bullet} \delta_{\geq n}(X^\bullet)[1]$$

in $K^*(\mathcal{A})$, where $\omega_n^p = 0$, for $p \neq n-1$; and $\omega_n^p = -d^p$, for $p = n-1$.

Proof. First, we claim that, for any $X^\bullet \in C^*(\mathcal{A})$ and $n \in \mathbb{Z}$, there exists in $K^*(\mathcal{A})$ an isomorphism

$$\bar{q}_n^\bullet : C^\bullet_{\mu_n} \to \delta_{<n}(X^\bullet),$$

where $C^\bullet_{\mu_n}$ is the mapping cone of the truncation $\mu_n^\bullet : \delta_{\geq n}(X^\bullet) \to X^\bullet$.

In fact, let $q_n^\bullet : C^\bullet_{\mu_n} \to \delta_{<n}(X^\bullet)$ be the following morphism:

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] \arrow[d, equal] & X^{n-2} \arrow[r, "\binom{d^{n-2}}{0}"] \arrow[d, equal] & X^{n-1} \oplus X^n \arrow[r, "d^{n-1}_{C_{\mu_n}}"] \arrow[d, "(1,0)"] & X^n \oplus X^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] & X^{n-2} \arrow[r, "d^{n-2}"] & X^{n-1} \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Then consider this morphism $t_n^\bullet : \delta_{<n}(X^\bullet) \to C^\bullet_{\mu_n}$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] \arrow[d, equal] & X^{n-2} \arrow[r, "d^{n-2}"] \arrow[d, equal] & X^{n-1} \arrow[r] \arrow[d, "\binom{1}{-d^{n-1}}"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] & X^{n-2} \arrow[r, "\binom{d^{n-2}}{0}"] & X^{n-1} \oplus X^n \arrow[r, "d^{n-1}_{C_{\mu_n}}"] & X^n \oplus X^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

We get a morphism $t_n^\bullet q_n^\bullet - 1_{C_{\mu_n}} : C^\bullet_{\mu_n} \to C^\bullet_{\mu_n}$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"] \arrow[d, "0"] & X^{n-2} \arrow[r, "\binom{d^{n-2}}{0}"] \arrow[d, "0"] & X^{n-1} \oplus X^n \arrow[r, "d^{n-1}_{C_{\mu_n}}"] \arrow[d, "\begin{pmatrix} 0 & 0 \\ -d^{n-1} & -1 \end{pmatrix}"] & X^n \oplus X^{n+1} \arrow[r] \arrow[d, "\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}"] & \cdots \\
\cdots \arrow[r] & X^{n-3} \arrow[r, "d^{n-3}"'] & X^{n-2} \arrow[r, "\binom{d^{n-2}}{0}"'] & X^{n-1} \oplus X^n \arrow[r, "d^{n-1}_{C_{\mu_n}}"'] & X^n \oplus X^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now we define a family of morphisms $(h^p)_{p\in\mathbb{Z}}$ in $\mathcal{A}$, where $h^p : C^p_{\mu_n} \to C^{p-1}_{\mu_n}$ such that $h^p = 0$, for $p < n$; and

$$h^p = \begin{bmatrix} 0 & 0 \\ -1_{X^p} & 0 \end{bmatrix}, \text{for } p \geq n.$$

Thus, we know

$$h^n d^{n-1}_{C_n} = \begin{bmatrix} 0 & 0 \\ -d^{n-1} & -1 \end{bmatrix}$$

and for all $p \geq n$ we have

$$h^p d^{p-1}_{C_n} + d^p_{C_{\mu_n}} h^{p+1} = -1_{C^p_n}.$$

Hence, we get that $t_n q_n - 1_{C_{\mu_n}}$ is homotopic to zero. Therefore, $\bar{t}^\bullet_n\,\bar{q}^\bullet_n = 1_{C_{\mu_n}}$ in $K^*(\mathcal{A})$. Analogously, we can get $\bar{q}^\bullet_n\,\bar{t}^\bullet_n = 1_{C_{\mu_n}}$ in $K^*(\mathcal{A})$. Thus, the claim is true.

Moreover, we know that there is an exact triangle

$$\delta_{\geq n}(X^\bullet) \xrightarrow{\bar{\mu}^\bullet_n} X^\bullet \xrightarrow{\bar{i}^\bullet_n} C^\bullet{}_{\mu_n} \xrightarrow{\bar{p}^\bullet_n} \delta_{\geq n}(X^\bullet)[1] \in K^*(\mathcal{A}),$$

where $i^\bullet_n$ is the canonical injection and $p^\bullet_n$ is the canonical projection. Since $\bar{\omega}^\bullet_n = \bar{p}^\bullet_n\,\bar{t}^\bullet_n$ and $\bar{t}^\bullet_n\,\bar{q}^\bullet_n = 1$ in $K^*(\mathcal{A})$, we get $\bar{\omega}^\bullet_n\,\bar{q}^\bullet_n = \bar{p}^\bullet_n\,\bar{t}^\bullet_n\,\bar{q}^\bullet_n = \bar{p}^\bullet_n$. And knowing $\bar{q}^\bullet_n\,\bar{i}^\bullet_n = \bar{\pi}^\bullet_n$, we can obtain a commutative diagram

$$\begin{tikzcd}
\delta_{\geq n}(X^\bullet) \arrow[r, "\bar{\mu}^\bullet_n"] \arrow[d, equal] & X^\bullet \arrow[r, "\bar{i}^\bullet_n"] \arrow[d, equal] & C^\bullet{}_{\mu^\bullet_n} \arrow[r, "\bar{p}^\bullet_n"] \arrow[d, "\bar{q}^\bullet_n"] & \delta_{\geq n}(X^\bullet)[1] \arrow[d, equal] \\
\delta_{\geq n}(X^\bullet) \arrow[r, "\bar{\mu}^\bullet_n"] & X^\bullet \arrow[r, "\bar{\pi}^\bullet_n"] & \delta_{<n}(X^\bullet) \arrow[r, "\bar{\omega}^\bullet_n"] & \delta_{\geq n}(X^\bullet)[1]
\end{tikzcd}$$

Since $\bar{q}^\bullet_n$ is an isomorphism in $K^*(\mathcal{A})$, we know

$$\delta_{\geq n}(X^\bullet) \xrightarrow{\bar{\mu}^\bullet_n} X^\bullet \xrightarrow{\bar{\pi}^\bullet_n} \delta_{<n}(X^\bullet) \xrightarrow{\bar{\omega}^\bullet_n} \delta_{\geq n}(X^\bullet)[1]$$

is an exact triangle in $K^*(\mathcal{A})$. The proof of the theorem is completed.

For any $X^\bullet \in C^b(\mathcal{A})$, we denote by $w(X^\bullet)$ the number of the non-zero objects in $X^\bullet$.

Now we introduce other two kinds of complexe truncation. Consider a complex

$$X^\bullet: \quad \cdots \longrightarrow X^{p-1} \xrightarrow{d^{p-1}} X^p \xrightarrow{d^p} X^{p+1} \longrightarrow \cdots$$

in $C^*(\mathcal{A})$. For each $n \in \mathbb{Z}$, we define

$$\tau_{\geq n}(X^\bullet): \quad \cdots \longrightarrow 0 \longrightarrow \mathrm{coker} d^{n-1} \longrightarrow X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \longrightarrow \cdots;$$

$$\tau_{\leq n}(X^\bullet): \quad \cdots \longrightarrow X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \longrightarrow \ker d^n \longrightarrow 0 \longrightarrow \cdots.$$

The following result is well known. Here we include a proof for self-completeness.

4.1.19 Lemma. *Let $\mathcal{A}$ be a full additive subcategory of an abelian category $\mathscr{A}$, and let $X^\cdot$ be a complex in $D(\mathcal{A})$. If $X^\cdot$ has bounded cohomology, then it is isomorphic to a bounded complex $Y^\bullet$ in $D(\mathcal{A})$.*

Proof. We may assume that there exists an integer $n > 0$ such that $H^p(X^\bullet) = 0$ in case $p \notin [0, n]$. Write $Y^\bullet = \tau_{\leq n}(\tau_{\geq 0}(X^\bullet))$ is in $C^b(\mathcal{A})$. It is easy to see $H^p(Y^\bullet) = H^p(X^\bullet)$, for $p \in \mathbb{Z}$. The proof of the lemma is completed.

4.2 Auslander-Reiten Triangles in Derived Categories

In this section, let A be an R-algebra, where R is a commutative ring. We denote by Mod$A$ the category of all left A-modules, and by mod$A$ the category of finitely generated left A-modules. Similarly, ModA^{op} stands for the category of all right A-modules, and A-mod for the category of finitely generated right A-modules. We fix an injective cogenerator I for Mod$R$. Then we have two contravariant exact functors such that

$$D: \mathrm{Mod}A \to A\text{-Mod}: M \mapsto \mathrm{Hom}_R(M,\ I)$$

and

$$D : A\text{-Mod} \to \mathrm{Mod}A : N \mapsto \mathrm{Hom}_R(N,\ I).$$

This yields the Nakayama functor

$$\nu = D\mathrm{Hom}_A(-, A) : \mathrm{Mod}A \to \mathrm{Mod}A : M \mapsto D\mathrm{Hom}_A(M,\ A).$$

4.2.1 DEFINITION. Let A be an R-algebra, and let P be a left A-module. Consider $u_i \in P$ and $\varphi_i \in \mathrm{Hom}_A(P, A)$ with $i \in I$. One says that $\{u_i; \varphi_i\}_{i\in I}$ is a *projective basis* of P if $u = \sum_{i\in I} \varphi_i(u)u_i$, for any $u \in P$, where $\varphi_i(u) = 0$ for all but finitely many $i \in I$.

The following result is well known; see, for example, [25]. For the convenience of the reader, we present here the proof.

4.2.2 LEMMA. *Let A be an R-algebra. A left A-module P is* (*finitely generated*) *projective if and only if it has a* (*finite*) *projective basis.*

Proof. Assume that P is projective. There exists a free left A-module F and a surjective A-linear map $\psi : F \to P$. By the projectivity of P, there is an A-linear map $\varphi : P \to F$ such that $\psi\varphi = 1$. Let $\{e_i : i \in I\}$ be a basis of F, and define $u_i = \psi(e_i)$. If $u \in P$, then there is a unique expression $\varphi(u) = \sum_i a_i e_i$, where $a_i \in A$ and almost all $a_i = 0$. Define $\varphi_i(u) = a_i$. Of course, given u, we have $\varphi_i(u) = 0$ for almost all i. Finally,

$$u = \psi\varphi(u) = \psi(\textstyle\sum_i a_i e_i) = \sum_i a_i\psi(e_i) = \sum_i(\varphi(u))\psi(e_i) = \sum_i(\varphi(u))u_i.$$

Since ψ is surjective, P is generated by $\{u_i : i \in I\}$.

Conversely, given $\{u_i \in P\}$ and a family of A-linear maps $(\varphi_i : P \to A)_{i\in I}$ as a projective basis $\{u_i, \varphi_i\}_{i\in I}$ of P, define F to be the free A-module with basis $\{e_i : i \in I\}$, and define an A-linear map $\psi : F \to P$ by $\psi : e_i \to u_i$. It suffices to find an A-linear map $\varphi : P \to F$ with $\varphi\psi = 1$, for then P is (isomorphic to) a direct summand of F, and hence P is projective. Define φ by $\varphi(u) = \sum_i(\varphi_i(u))e_i$, for $u \in P$. By definition of projective basis, the sum is finite, and so φ is well defined. By definition of projective basis, for each $u \in P$, we have

$$\psi\varphi(u) = \psi(\textstyle\sum_i(\varphi_i(u)e_i)) = \sum_i(\varphi_i(u))\psi(e_i) = \sum(\varphi(u))u_i = u.$$

That is, $\psi\varphi = 1$. The proof of the lemma is completed.

4.2.3 Lemma. *Let A be an R-algebra, and let P be a left A-module. If $\{u_i; \varphi_i\}_{1\le i\le n}$ is a finite projective basis of P, then $\{\varphi_i; \hat{u}_i\}_{1\le i\le n}$ is a projective basis of* $\mathrm{Hom}_A(P, A)$*, where*

$$\hat{u}_i : \mathrm{Hom}_A(P, A) \to A : \phi \mapsto \phi(u_i).$$

Proof. We know that $\mathrm{Hom}_A(P, A)$ is a right A-module such that, for any $v \in P$ and $a \in A$,

$$(\varphi_i a)(v) = \varphi_i(v)\, a,\ 1 \le i \le n.$$

Therefore, for any $\phi \in \mathrm{Hom}_A(P, A), v \in P$, we have

$$\begin{array}{rcl} (\sum_{i=1}^n \varphi_i\, \hat{u}_i(\phi))(v) & = & (\sum_{i=1}^n \varphi_i \phi(u_i))(v) \\ & = & \sum_{i=1}^n \left(\varphi_i \phi(u_i)\right)(v) \\ & = & \sum_{i=1}^n \varphi_i(v)\phi(u_i) \\ & = & \phi(\sum_{i=1}^n \varphi_i(v) u_i) \\ & = & \phi(v). \end{array}$$

That is, $\phi = \sum_{i=1}^n \varphi_i\, \hat{u}_i(\phi)$. The proof of the lemma is completed.

Remark. The above results say in particular that if P is a finitely generated projective left A-module, then $\mathrm{Hom}_A(P, A)$ is a finitely generated projective right A-module. Observe that the result fails if the projective basis is infinite.

The following well-known result is usually proved by applying adjoint isomorphisms. Here, we include a new elementary proof.

4.2.4 Theorem. *Let A be an R-algebra, and let P be a finitely generated projective A-module. For any $X \in \mathrm{Mod}A$, there exists an R-linear isomorphism*

$$\alpha_{P,X} : D\mathrm{Hom}_A(P, X) \to \mathrm{Hom}_A(X, \nu P),$$

which is functorial in P and X.

Proof. By Lemma 4.2.2, P has a finite projective basis $\{u_1, \ldots, u_n; \varphi_1, \ldots, \varphi_n\}$. For any $h \in \mathrm{Hom}_A(P, A)$ and $x \in X$, we define a map

$$h_x : P \to X : u \mapsto h(u)x,$$

which is is A-linear. Indeed, for any $a \in A, u \in P$, we have

$$h_x(au) = h(au)x = (ah(u))x = a(h(u)x) = ah_x(u).$$

Now, we define

$$\alpha_{P,X} : D\mathrm{Hom}_A(P, X) \to \mathrm{Hom}_A(X, \nu P) : \theta \mapsto (x \mapsto (h \mapsto \theta(h_x)))$$

and

$$\beta_{P,X} : \mathrm{Hom}_A(X, \nu P) \to D\mathrm{Hom}_A(P, X) : f \mapsto (g \mapsto \sum\nolimits_{i=1}^{n} (f(g(u_i))(\varphi_i)).$$

Since $\mathrm{Hom}_A(P, A)$ is a right A-module, $\nu P = \mathrm{Hom}_R(\mathrm{Hom}_A(P, A), I)$ is a left A-module such that $(a\psi)(\varphi) = \psi(\varphi a)$, for any $a \in A, \psi \in \nu P$, and $\varphi \in \mathrm{Hom}_A(P, A)$.

We first claim that $\alpha_{P,X}\beta_{P,X} = 1$. Indeed, for each $f \in \mathrm{Hom}_A(X, \nu P)$, we verify that $(\alpha_{P,X}(\beta_{P,X}(f)))(x) = f(x)$. Indeed, for $h \in \mathrm{Hom}_A(P, A)$, by definition, we have

$$\begin{array}{rcl} [(\alpha_{P,X}(\beta_{P,X}(f)))(x)](h) & = & \beta_{P,X}(f)(h_x) \\ & = & \sum_{i=1}^{n} (f(h_x(u_i)))(\varphi_i) \\ & = & \sum_{i=1}^{n} f(h(u_i)x)(\varphi_i) \\ & = & \sum_{i=1}^{n} (h(u_i)f(x))(\varphi_i) \\ & = & \sum_{i=1}^{n} f(x)(\varphi_i h(u_i)) \\ & = & f(x)(\sum_{i=1}^{n} \varphi_i h(u_i)) \\ & = & f(x)\,(\sum_{i=1}^{n} \varphi_i\, \hat{u}_i(h)) \\ & = & (f(x))(h), \end{array}$$

where the last equation follows from Lemma 4.2.3. This establishes our first claim.

Next, we claim that $\beta_{P,X}\alpha_{P,X} = 1$. For this end, consider $\theta \in D\mathrm{Hom}_A(P, X)$ and $g \in \mathrm{Hom}_A(P, X)$. By definition, we have

$$\begin{array}{rcl}(\beta_{P,X}(\alpha_{_{P,X}}(\theta)))(g) & = & \sum_{i=1}^{n}\left((\alpha_{_{P,X}}(\theta)(g(u_i))\right)(\varphi_i)\\ & = & \sum_{i=1}^{n}\theta((\varphi_i)_{\,g(u_i)})\\ & = & \theta\left(\sum_{i=1}^{n}(\varphi_i)_{\,g(u_i)}\right)\\ & = & \theta(g),\end{array}$$

where the last equation can be deduced as follows. For any $u \in P$, we have

$$\begin{array}{rcl}\sum_{i=1}^{n}((\varphi_i)_{\,g(u_i)})(u) & = & \sum_{i=1}^{n}\varphi_i(u)g(u_i)\\ & = & \sum_{i=1}^{n}g(\varphi_i(u)u_i)\\ & = & g\left(\sum_{i=1}^{n}\varphi_i(u)u_i\right)\\ & = & g(u).\end{array}$$

Therefore, our second claim also holds. That is, $\alpha_{_{P,X}}$ is an isomorphism. It remains to verify the naturality of $\alpha_{_{P,X}}$ in P and X. First, let $t : X \to Y$ be a morphism in Mod$A$. Consider the following diagram

$$\begin{tikzcd} D\mathrm{Hom}_A(P,Y) \arrow[r, "\alpha_{P,Y}"] \arrow[d, "D\mathrm{Hom}_A(P,t)"'] & \mathrm{Hom}_A(Y,\nu P) \arrow[d, "\mathrm{Hom}_A(t,\nu P)"] \\ D\mathrm{Hom}_A(P,X) \arrow[r, "\alpha_{P,X}"] & \mathrm{Hom}_A(X,\nu P). \end{tikzcd}$$

Let $\theta \in D\mathrm{Hom}_A(P,Y)$, for any $x \in X, h \in \mathrm{Hom}_A(P,A)$, we have

$$(\mathrm{Hom}_A(t,\nu P)\circ\alpha_{_{P,Y}}(\theta))(x)(h) = \alpha_{_{P,Y}}(\theta)(t)(x)(h) = \theta(h_{t(x)}).$$

On the other side,

$$(\alpha_{_{P,X}}\circ D\mathrm{Hom}_A(P,t)(\theta))(x)(h) = \theta(th_x).$$

Moreover, for any $u \in P$ we have

$$h_{t(x)}(u) = h(u)t(x) = t((h(u))x) = th_x(u).$$

Therefore, the above diagram commutes. Similarly, let $P, Q \in \mathcal{P}_A$ and a morphism $s : P \to Q$. Consider the following diagram

$$\begin{tikzcd} D\mathrm{Hom}_A(P,X) \arrow[r, "\alpha_{P,X}"] \arrow[d, "D\mathrm{Hom}(s,X)"'] & \mathrm{Hom}_A(X,\nu P) \arrow[d, "\mathrm{Hom}(X,\nu s)"] \\ D\mathrm{Hom}_A(Q,X) \arrow[r, "\alpha_{Q,X}"] & \mathrm{Hom}_A(X,\nu Q). \end{tikzcd}$$

Let $\theta \in D\mathrm{Hom}_A(P, X)$. For any $x \in X, h \in \mathrm{Hom}_A(Q, A)$, we have

$$(\mathrm{Hom}_A(X, \nu s) \circ \alpha_{P,X}(\theta))(x)(h) = \alpha_{P,X}(\theta)(x)(hs) = \theta((hs)_x).$$

On the other side,

$$(\alpha_{Q,X} \circ D\mathrm{Hom}_A(s, X)(\theta))(x)(h) = \alpha_{Q,X}(\theta')(x)(h) = \theta'(h_x) = \theta(h_x s),$$

where $\theta' = D\mathrm{Hom}_A(s, X)(\theta)$. For any $u \in Q$, we have

$$h_x t(u) = h(t(u))x = (ht)(u)x = (ht)_x(u).$$

Therefore, the above diagram commutes. The proof of the theorem is completed.

4.2.5 Corollary. *Let A be an R-algebra. If P is a finitely generated projective A-module, then νP is an injective A-module.*
Proof. In view of Theorem 4.2.4, we see that $\mathrm{Hom}_A(-, \nu P) \cong D\mathrm{Hom}_A(P, -)$. Since both $\mathrm{Hom}_A(P, -)$ and D are exact, so is $\mathrm{Hom}_A(-, \nu P)$. That is, νP is injective. The proof of the corollary is completed.

Let $\mathcal{P}_A$ denote the category of finitely generated projective left A-modules. The following result is probably well know. Here we give a detailed proof.

4.2.6 Lemma. *Let A be an R-algebra. If $P^\bullet \in C^b(\mathcal{P}_A)$ then, for any complex $X^\bullet \in C^b(\mathrm{Mod}A)$, there is a morphism*

$$\begin{array}{rrcl} \beta_{P^\bullet,X^\bullet}: & \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) & \longrightarrow & D\mathrm{Hom}_{C^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet) \\ & \xi^\bullet & \longrightarrow & \beta_{P^\bullet,X^\bullet}(\xi^\bullet) : \eta^\bullet \mapsto \sum_{i\in\mathbb{Z}} (-1)^i \beta_{P^i X^i}(\xi^i)(\eta^i). \end{array}$$

(1) *If $\xi^\bullet$ or $\eta^\bullet$ is null-homotopic, then $\beta_{P^\bullet,X^\bullet}(\xi^\bullet)(\eta^\bullet) = 0$.*

(2) *The map $\beta_{P^\bullet,X^\bullet}$ is functorial in $P^\bullet$ and $X^\bullet$.*

Proof. For proving (1), we may assume that there exists an integer $n > 0$ such that $P^i = 0$ and $X^i = 0$ in case $i \notin [0, n]$. Let $\xi^\bullet \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet)$ and $\eta^\bullet \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet)$. Suppose first that $\xi^\bullet$ is null-homotopic. Let

$h^i : X^i \to \nu(P^{i-1})$, $1 \leq i \leq n$, be morphisms such that $\xi^0 = h^1 d_X^0$, and $\xi^i = h^{i+1} d_X^i + \nu(d_P^{i-1}) h^i$, for $1 \leq i < n$, and $\xi^n = \nu(d_P^{n-1}) h^n$. This yields

$$\begin{aligned}
& \beta_{P^\bullet, X^\bullet}(\xi^\bullet)(\eta^\cdot) \\
= \; & \textstyle\sum_{i=0}^{n} (-1)^i \beta_{P^i, X^i}(\xi^i)(\eta^i) \\
= \; & \beta_{P^0, X^0}(h^1 d_X^0)(\eta^0) + (-1)^n \beta_{P^n, X^n}(\nu(d_P^{n-1}) h^n)(\eta^n) \\
& + \textstyle\sum_{i=1}^{n-1} (-1)^i \beta_{P^i X^i}(h^{i+1} d_X^i + \nu(d_P^{i-1}) h^i)(\eta^i) \\
= \; & \beta_{P^0, X^0}(h^1 d_X^0)(\eta^0) + \textstyle\sum_{i=1}^{n-1} (-1)^i \beta_{P^i, X^i}(h^{i+1} d_X^i)(\eta^i) \\
& + \textstyle\sum_{i=1}^{n-1} (-1)^i \beta_{P^i, X^i}(\nu(d_P^{i-1}) h^i)(\eta^i) + (-1)^n \beta_{P^n, X^n}(\nu(d_P^{n-1}) h^n)(\eta^n) \\
= \; & \textstyle\sum_{i=0}^{n-1} (-1)^i \beta_{P^i, X^i}(h^{i+1} d_X^i)(\eta^i) + \textstyle\sum_{i=1}^{n} (-1)^i \beta_{P^i, X^i}(\nu(d_P^{i-1}) h^i)(\eta^i) \\
= \; & \textstyle\sum_{i=1}^{n} (-1)^{i-1} \beta_{P^{i-1}, X^{i-1}}(h^i d_X^{i-1})(\eta^{i-1}) + \textstyle\sum_{i=1}^{n} (-1)^i \beta_{P^i, X^i}(\nu(d_P^{i-1}) h^i)(\eta^i) \\
= \; & \textstyle\sum_{i=1}^{n} (-1)^{i-1} \beta_{P^{i-1}, X^i}(h^i)(d_X^{i-1} \eta^{i-1}) + \textstyle\sum_{i=1}^{n} (-1)^i \beta_{P^{i-1}, X^i}(h^i)(\eta^i d_P^{i-1}) \\
= \; & \textstyle\sum_{i=1}^{n} (-1)^{i-1} \beta_{P^{i-1}, X^i}(h^i)(\eta^i d_P^{i-1}) + \textstyle\sum_{i=1}^{n} (-1)^i \beta_{P^{i-1}, X^i}(h^i)(\eta^i d_P^{i-1}) \\
= \; & 0,
\end{aligned}$$

where the sixth equation follows from the naturality of $\beta_{P,X}$ in P and in X. Similarly, one can show that $\beta_{P^\bullet, X^\bullet}(\xi^\bullet)(\eta^\cdot) = 0$ if $\eta^\cdot$ is null-homotopic.

(2) Fix $X^\bullet \in C^b(\mathrm{Mod} A)$. Let $g : P^\bullet \to Q^\bullet$ be a morphism in $C^b(\mathcal{P}_A)$. For any morphisms $\xi \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet)$ and $\eta \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(Q^\bullet, X^\bullet)$, we have

$$(D\mathrm{Hom}_{C^b(\mathrm{Mod}A)}(g, X^\bullet) \circ \beta_{P^\bullet, X^\bullet}(\xi))(\eta) \;=\; \textstyle\sum (-1)^i \beta_{P^i X^i}(\xi^i)(\eta^i g^i)$$

and

$$\begin{aligned}
(\beta_{Q^\bullet, X^\bullet} \circ \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(X^\bullet, \nu(g))(\xi))(\eta) \;&=\; \beta_{Q^\bullet, X^\bullet}(\nu(g)\xi)(\eta) \\
&=\; \textstyle\sum (-1)^i \beta_{Q^i, X^i}(\nu(g^i)\xi^i)(\eta^i).
\end{aligned}$$

Since $\beta_{P,X}$ is functorial in P and X, for each i, we have

$$\beta_{P^i, X^i}(\xi^i)(\eta^i g^i) = \beta_{Q^i, X^i}(\nu(g^i)\xi^i)(\eta^i).$$

Thus, we get a commutative diagram

$$\begin{tikzcd}
(X^\bullet, \nu P^\bullet) \arrow[r, "\beta_{P^\bullet,X^\bullet}"] \arrow[d, "(X^\cdot{,}\nu(g))"'] & D(P^\bullet, X^\bullet) \arrow[d, "D(g{,}X^\bullet)"] \\
(X^\bullet, \nu Q^\bullet) \arrow[r, "\beta_{Q^\bullet,X^\bullet}"] & D(Q^\bullet, X^\bullet),
\end{tikzcd}$$

where $(-,-)$ denotes $\mathrm{Hom}_{C^b(\mathrm{Mod}A)}(-,-)$.

Fix $P^\bullet \in C^b(\mathcal{P}_A)$. Let $f : X^\bullet \to Y^\bullet$ be a morphism in $C^b(\mathrm{Mod}A)$. Consider this diagram

$$\begin{tikzcd}
(Y^\bullet, \nu P^\bullet) \arrow[r, "\beta_{P^\bullet,Y^\bullet}"] \arrow[d, "(f{,}\nu P^\bullet)"'] & D(P^\bullet, Y^\bullet) \arrow[d, "D(P^\bullet{,}f)"] \\
(X^\bullet, \nu P^\bullet) \arrow[r, "\beta_{P^\bullet,X^\bullet}"] & D(P^\bullet, X^\bullet),
\end{tikzcd}$$

where $(-,-)$ denotes $\mathrm{Hom}_{C^b(\mathrm{Mod}A)}(-,-)$. For any $\xi \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(Y^\bullet, \nu P^\bullet)$ and $\eta \in \mathrm{Hom}_{C^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet)$, we have

$$(D\mathrm{Hom}_{C^b(\mathrm{Mod}(A)}(P^\bullet, f) \circ \beta_{P^\bullet,Y^\bullet}(\xi))(\eta) \quad = \quad \sum(-1)^i \beta_{P^i,Y^i}(\xi^i)(f^i\eta^i)$$

and

$$\begin{aligned}
(\beta_{P^\bullet,X^\bullet} \circ \mathrm{Hom}_{C^b(\mathrm{Mod}(A)}(f, \nu P^\bullet)(\xi)))(\eta) \quad &= \quad \beta_{P^\bullet,X^\bullet}(\xi f)(\eta) \\
&= \quad \sum(-1)^i \beta_{P^i,X^i}(\xi^i f^i)(\eta^i).
\end{aligned}$$

Using again the naturality of $\beta_{P,X}$, for each i, we know that

$$\beta_{P^i,Y^i}(\xi^i)(f^i\eta^i) = \beta_{P^i,X^i}(\xi^i f^i)(\eta^i).$$

Thus, the above diagram commutes. The proof of the lemma is completed.

Let $\mathcal{P}_A$ be the full subcategory of mod$A$ of projective A-modules and $\mathcal{I}_\mathcal{A}$ the full subcategory of art(A) of injective modules.

4.2.7 Proposition. *Let A be a noetherian R-algebra with R being complete local noetherian. Then the Nakayama functor induces an equivalence*

$$\nu : \mathcal{P}_A \longrightarrow \mathcal{I}_A : P \mapsto D\mathrm{Hom}_A(P, A).$$

Proof. Suppose that P, Q are two modules in $\mathcal{P}_A$. By Theorem 4.2.4, we have

$$\mathrm{Hom}_A(\nu Q, \nu P) \cong D\mathrm{Hom}_A(P, \nu Q) \cong D^2\mathrm{Hom}_A(Q, P).$$

Observing that $\mathrm{Hom}_A(Q, P)$ is a finitely generated R-module, we obtain

$$\mathrm{Hom}_A(Q, P) \cong D^2\mathrm{Hom}_A(Q, P) \cong \mathrm{Hom}_A(\nu Q, \nu P).$$

That is, ν is fully faithful. Now, suppose that I is in $\mathcal{I}_A$. In particular, $I \in \mathrm{art}(A)$. Hence, $I = I_1 \oplus \cdots \oplus I_n$, where I_i is indecomposable injective, $i = 1, \ldots, n$. Since I_i is artinian and indecomposable, it has an essential simple socle S_i, and hence, the inclusion $f_i : S_i \to I_i$ is an injective envelope. Observing that $D(S_i)$ is a simple module in $\mathrm{noe}(A)$, we have a projective cover $g_i : P_i \to D(S_i)$. Applying the duality D, we obtain an injective envelope $D(g_i) : D^2(S_i) \to D(P_i)$ in $\mathrm{art}(A)$. Since $D^2(S_i) \cong S_i$, we know $D(P_i) \cong I_i$. Thus, ν is dense. The proof of the proposition is completed.

4.2.8 PROPOSITION. *Let $\mathcal{A}$ be an additive R-category in which idempotents split, where R is a complete local noetherian commutative ring. If $\mathcal{A}$ is Hom-finite, then it is Krull-Schmidt.*

Proof. Assume that X is an object in $\mathcal{A}$. By the assumption, $\mathrm{End}_{\mathcal{A}}(X)$ is finitely generated as an R-module, that is, $\mathrm{End}_{\mathcal{A}}(X)$ is noetherian R-algebra. Since R is complete local noetherian, $\mathrm{End}_{\mathcal{A}}(X)$ is semiperfect; see, for example, [4, Section 5]. Therefore, $\mathcal{A}$ is Krull-Schmidt; see, for example, [22, (1.1)]. The proof of the proposition is completed.

4.2.9 PROPOSITION. *Let A be a noetherian R-algebra with R being complete local noetherian.*

(1) *The Nakayama equivalence $\nu : \mathcal{P}_A \to \mathcal{I}_A$ induces a triangle equivalence $\nu : K^b(\mathcal{P}_A) \to K^b(\mathcal{I}_A)$.*

(2) *$K^b(\mathcal{P}_A)$ and $K^b(\mathcal{I}_A)$ are Krull-Schmidt.*

Proof. The statement (1) is an immediate consequence of Proposition 4.2.7. For proving (2), we only need to prove that $K^b(\mathcal{P}_A)$ is Krull-Schmidt. Since

$\mathcal{P}_A$ is Hom-finite and R is noetherian, so is $K^b(\mathcal{P}_A)$. By Proposition 4.2.8, it remains to show that the idempotents in $K^b(\mathcal{P}_A)$ is split. Firstly, by Proposition 1.2.2, the idempotents in mod$A$ split and since $\mathcal{P}_A$ is closed under taking direct summands, the idempotents in $\mathcal{P}_A$ split.

Let $P^\bullet$ be a complex in $K^b(\mathcal{P}_A)$. We claim that the idempotents of $\mathrm{End}_{K^b(\mathcal{P}_A)}(P^\bullet)$ split in $K^b(\mathcal{P}_A)$. Suppose that $w(P^\bullet) = 1$. By our previous remark, the claim is true. Assume now that $w(P^\bullet) = n > 1$. We may assume that $P^i = 0$ for any $i \notin [1, n]$. By Theorem 4.1.18, there exists an exact triangle

$$\delta_{\geq n}(P^\bullet) \xrightarrow{\bar{\mu}_n} P^\bullet \xrightarrow{\bar{\pi}_n} \delta_{<n}(P^\bullet) \longrightarrow \delta_{\geq n}(P^\bullet)[1],$$

where $w(\delta_{\geq n}(P^\bullet)) \leq 1$ and $w(\delta_{<n}(P^\bullet)) < n$. Let e be an idempotent in $\mathrm{End}_{K^b(\mathcal{P}_A)}(P^\bullet)$, and let u be the restriction of e to $\delta_{\geq n}(P^\bullet)$, and v the restriction of e to $\delta_{<n}(P^\bullet)$. Then we obtain a commutative diagram

$$\begin{tikzcd}
\delta_{\geq n}(P^\bullet) \arrow[r, "\mu_n"] \arrow[d, "u"] & P^\bullet \arrow[r, "\pi_n"] \arrow[d, "e"] & \delta_{<n}(P^\bullet) \arrow[r] \arrow[d, "v"] & \delta_{\geq n}(P^\bullet)[1] \arrow[d, "u[1]"] \\
\delta_{\geq n}(P^\bullet) \arrow[r, "\mu_n"] & P^\bullet \arrow[r, "\pi_n"] & \delta_{<n}(P^\bullet) \arrow[r] & \delta_{\geq n}(P^\bullet)[1]
\end{tikzcd}$$

By the induction hypothesis, u and v split in $K^b(\mathcal{P}_A)$, and hence, so does e; see [20, 2.3]. The proof of the proposition is completed.

The following result is well known; see, for example, [27, (10.4.7)].

4.2.10 Lemma. *Let $\mathscr{A}$ be an abelian category, and let $X^\bullet, Y^\bullet$ be complexes over $\mathscr{A}$. If $X^\bullet$ is bounded-above of projective objects or $Y^\bullet$ is bounded-below of injective objects, then the localization functor $L_{\mathscr{A}} : K^b(\mathscr{A}) \to D^b(\mathscr{A})$ induces an isomorphism*

$$L_{X^\bullet, Y^\bullet} : \mathrm{Hom}_{K^b(\mathscr{A})}(X^\bullet, Y^\bullet) \to \mathrm{Hom}_{D^b(\mathscr{A})}(X^\bullet, Y^\bullet) : \bar{f}^\bullet \to \tilde{f}^\bullet.$$

The above result enables us to obtain the following important isomorphism theorem.

4.2.11 PROPOSITION. *Let A be a noetherian R-algebra with R being complete local noetherian, and let $X^\bullet, P^\bullet$ be complexes in $D^b(\mathrm{Mod}A)$. If $P^\bullet$ is of projective objects, then there exists an R-linear isomorphism*

$$\beta_{P^\bullet,X^\bullet} : \mathrm{Hom}_{D^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet)$$

in $D^b(\mathrm{Mod}A)$, *which is functorial in $P^\bullet$ and $X^\bullet$.*

Proof. By Lemma 4.2.6, we have an R-linear morphism

$$\begin{array}{rrcl} \beta_{P^\bullet,X^\bullet} : & \mathrm{Hom}_{K^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) & \longrightarrow & D\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet) \\ & \bar{\xi}^\bullet & \mapsto & \left(\bar{\eta}^\bullet \mapsto \sum_{i\in\mathbb{Z}} (-1)^i \beta_{P^i X^i}(\xi^i)(\eta^i)\right), \end{array}$$

which is natural in $P^\bullet$ and $X^\bullet$. We shall proceed by a double induction to show that $\beta_{P^\bullet,X^\bullet}$ is an isomorphism. First, we consider the case where $w(X^\bullet) = 1$. With no loss of generality, we may assume that the non-zero component is of degree zero. Moreover, assume that $w(P^\bullet) = 1$. If the non-zero component of $P^\bullet$ is of degree zero, since the canonical functor $P : \mathrm{Mod}A \to K^b(\mathrm{Mod}A)$ is fully faithful, $\beta_{P^\bullet,X^\bullet}$ is an isomorphism. Otherwise, both $\mathrm{Hom}_{K^b(\mathrm{mod}A)}(X^\bullet, P^\bullet) = 0$ and $D\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet) = 0$, and hence, $\beta_{P^\bullet,X^\bullet}$ is trivially an isomorphism. Assume now that $w(P^\bullet) = n > 1$. Since $P^\bullet \in K^b(\mathcal{P}_A)$, in view of Theorem 4.1.18, there is an exact triangle

$$P_1^\bullet \longrightarrow P^\bullet \longrightarrow P_2^\bullet \longrightarrow P_1^\bullet[1]$$

in $K^b(\mathcal{P}_A)$ with $w(P_1^\bullet) < n$ and $w(P_2^\bullet) \le 1$. In view of Proposition 4.2.9, we deduce that

$$\nu P_1^\bullet \longrightarrow \nu P^\bullet \longrightarrow \nu P_2^\bullet \longrightarrow \nu P_1^\bullet[1]$$

is an exact triangle in $K^b(\mathrm{Mod}A)$. Therefore, we have the following commutative diagram with exact rows,

$$\begin{tikzcd} (X^\bullet, \nu P_2^\bullet[-1]) \arrow[r] \arrow[d, "\beta_{P_2^\bullet[-1],X^\bullet}"] & (X^\bullet, \nu P_1^\bullet) \arrow[r] \arrow[d, "\beta_{P_1^\bullet X^\bullet}"] & (X^\bullet, \nu P^\bullet) \arrow[r] \arrow[d, "\beta_{P^\bullet,X^\bullet}"] & (X^\bullet, \nu P_2^\bullet) \arrow[r] \arrow[d, "\beta_{P_2^\bullet,X^\bullet}"] & (X^\bullet, \nu P_1^\bullet[1]) \arrow[d, "\beta_{P_1^\bullet[1],X^\bullet}"] \\ D(P_2[-1]^\bullet, X^\bullet) \arrow[r] & D(P_1^\bullet, X^\bullet) \arrow[r] & D(P^\bullet, X^\bullet) \arrow[r] & D(P_2^\bullet, X^\bullet) \arrow[r] & D(P_1^\bullet[1], X^\bullet) \end{tikzcd}$$

where $(-,-)$ denotes $\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(-,-)$. By the induction hypothesis, we know that $\beta_{P_2^\bullet[-1],X^\bullet}, \beta_{P_1^\bullet,X^\bullet}, \beta_{P_2^\bullet,X^\bullet}, \beta_{P_1^\bullet[1],X^\bullet}$ are isomorphisms, and so is $\beta_{P^\bullet,X^\bullet}$.

By the induction on $w(P^\bullet)$, we have shown that $\beta_{P^\bullet,X^\bullet}$ is an isomorphism in case $w(X^\bullet) = 1$.

Assume that $w(X^\bullet) = m > 1$. Using Theorem 4.1.18 again, we obtain an exact triangle

$$Z^\bullet \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet[1]$$

in $K^b(\mathrm{Mod}A)$ with $w(Z^\bullet) < m$ and $w(Y^\bullet) \leq 1$. Then we have the following commutative diagram with exact rows,

$$\begin{tikzcd}
(Z^\bullet[-1], \nu P^\bullet) \arrow[r] \arrow[d, "\beta_{P^\bullet, Z^\bullet[-1]}"] & (Y^\bullet, \nu P^\bullet) \arrow[r] \arrow[d, "\beta_{P^\bullet, Y^\bullet}"] & (X^\bullet, \nu P^\bullet) \arrow[r] \arrow[d, "\beta_{P^\bullet, X^\bullet}"] & (Z^\bullet, \nu P^\bullet) \arrow[r] \arrow[d, "\beta_{P^\bullet, Z^\bullet}"] & (Y^\bullet[1], \nu P^\bullet) \arrow[d, "\beta_{P^\bullet, Y^\bullet[1]}"] \\
D(P^\bullet, Z^\bullet[-1]) \arrow[r] & D(P^\bullet, Y^\bullet) \arrow[r] & D(P^\bullet, X^\bullet) \arrow[r] & D(P^\bullet, Z^\bullet) \arrow[r] & D(P^\bullet, Y^\bullet[1])
\end{tikzcd}$$

where $(-,-)$ denotes $\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(-,-)$. Using the induction hypothesis, we deduce that $\beta_{P^\bullet,X^\bullet}$ are an isomorphism. By Lemma 4.2.10, the isomorphism $\beta_{P^\bullet,X^\bullet} : \mathrm{Hom}_{K^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet)$ induces an isomorphism $\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) \to D\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet)$ which, by abuse of notation, is denoted again by $\beta_{P^\bullet,X^\bullet}$, making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) \arrow[r, "\beta_{P^\bullet,X^\bullet}"] \arrow[d, "L_{X^\bullet,\nu P^\bullet}"'] & D\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet) \arrow[d, "DL_{P^\bullet,X^\bullet}"] \\
\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(X^\bullet, \nu P^\bullet) \arrow[r, "\beta_{P^\bullet,X^\bullet}"] & D\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, X^\bullet).
\end{tikzcd}$$

Since $DL_{X^\bullet,\nu P^\bullet}$ and $DL_{P^\bullet,X^\bullet}$ are natural in $P^\bullet$ and $X^\bullet$, so is $\beta_{P^\bullet,X^\bullet}$. The proof of the proposition is completed.

The following result is well known for abelian categories with enough projective objects; see, for example, [11]. Here, we provide a detailed proof.

4.2.12 THEOREM. *Let A be a noetherian R-algebra with R being complete local noetherian. For any $X^\bullet \in C^-(\mathrm{mod}A)$, there exists a quasi-isomorphism $s^\bullet : P^\bullet \to X^\bullet$, where $P^\bullet \in C^-(\mathcal{P}_A)$.*

Proof. First of all, since A is noetherian, $\mathrm{mod}A$ is an abelian category. Suppose that $X^\bullet \in C^-(\mathrm{mod}A)$. We may assume that $X^i = 0$, for $i \geq 0$.

For each $i \geq 0$, set $P^i = 0$, and $d_P^i = 0 : P^i \to P^{i+1}$, and $s^i = 0 : P^i \to X^i$, and $j^i : K^i \to P^i$ the kernel of d_P^i. It is easy to see that the following conditions are satisfied for every $i \geq 0$.

(1) There exists a pullback diagram:

$$\begin{tikzcd} L^i \arrow[r, "u^i"] \arrow[d, "v^i"'] & K^{i+1} \arrow[d, "s^{i+1}\circ j^{i+1}"] \\ X^i \arrow[r, "d_X^i"] & X^{i+1}, \end{tikzcd}$$

where v^i is an epimorphism.

(2) $d_P^i = j^{i+1} \circ u^i \circ p^i$, where $p^i : P^i \to L^i$ is a projective cover of L^i.

(3) $s^i = v^i \circ p^t$.

Let $t \geq 0$ be an integer for which we have defined finitely generated projective A-modules P^i, and A-linear maps $d_P^i : P^i \to P^{i+1}$, and also A-linear epimorphisms $s^i : P^i \to X^i$, for all $i \geq t$, such that the conditions (1), (2), (3) as stated above are satisfied for every $i \geq t$.

Now, let $j^t : K^t \to P^t$ be the kernel of $d_P^t : P^t \to P^{t+1}$. Being abelian, mod$A$ admits a pullback diagram

$$(*) \qquad \begin{tikzcd} L^{t-1} \arrow[r, "u^{t-1}"] \arrow[d, "v^{t-1}"'] & K^t \arrow[d, "s^t\circ j^t"] \\ X^{t-1} \arrow[r, "d_X^{t-1}"] & X^t. \end{tikzcd}$$

We claim that v^{t-1} is an epimorphism. Indeed, by assumption, we have a pullback

$$\begin{tikzcd} L^t \arrow[r, "u^t"] \arrow[d, "v^t"'] & K^{t+1} \arrow[d, "s^{t+1}\circ j^{t+1}"] \\ X^t \arrow[r, "d_X^t"] & X^{t+1}. \end{tikzcd}$$

where $(K^{t+1},\ j^{t+1})$ is the kernel of $d_P^{t+1} : P^{t+1} \to P^{t+2}$. This yields a short exact sequence

$$0 \longrightarrow L^t \xrightarrow{\binom{u^t}{v^t}} K^{t+1} \oplus X^t \xrightarrow{(s^{t+1}j^{t+1}, -d_X^t)} X^{t+1}\ .$$

Now, let $x \in X^{t-1}$. Then $d_X^{t-1}(x) \in X^t$ such that

$$(s^{t+1}j^{t+1},\ -d_X^t)\begin{pmatrix} 0 \\ d_X^{t-1}(x) \end{pmatrix} = -d_X^t d_X^{t-1}(x) = 0,$$

there exists $y \in L^t$ such that

$$\begin{pmatrix} 0 \\ d_X^{t-1}(x) \end{pmatrix} = \begin{pmatrix} u^t \\ v^t \end{pmatrix}(y) = \begin{pmatrix} u^t(y) \\ v^t(y) \end{pmatrix}.$$

That is, $u^t(y) = 0$ and $v^t(y) = d_X^{t-1}(x)$.

By assumption, we have the following commutative diagram

$$\begin{tikzcd}
 & L^t \arrow[r, "u^t"] \arrow[dd, "v^t"] & K^{t+1} \arrow[d, "j^{t+1}"] & & \\
P^t \arrow[ur, "p^t"] \arrow[rr, dotted, "d_P^t"] \arrow[dr, dotted, "s^t"'] & & P^{t+1} \arrow[r, "d_P^{t+1}"] \arrow[d, "s^{t+1}"] & P^{t+2} \arrow[r] \arrow[d, "s^{t+2}"] & \cdots \\
\cdots \arrow[r] & X^t \arrow[r, "d_X^t"] & X^{t+1} \arrow[r, "d_X^{t+1}"] & X^{t+2} \arrow[r] & \cdots
\end{tikzcd}$$

where $p^t : P^t \to L^t$ is a projective cover of L^t, and $d_P^t = j^{t+1}u^t p^t,\ s^{t+1} = v^t p^t$. Since p^t is an epimorphism, there exists $z \in P^t$ such that $p^t(z) = y$. Thus, $d_P^t(z) = j^{t+1}u^t p^t(z) = j^{t+1}u^t(y) = 0$. Hence $z \in K^t$. And $d_X^{t-1}(x) = v^t(y) = v^t p^t(z) = s^t(z) = (s^t \circ j^t)(z)$.

In view of the pullback diagram $(*)$, we have a short exact sequence

$$0 \longrightarrow L^{t-1} \xrightarrow{\binom{u^{t-1}}{v^{t-1}}} K^t \oplus X^{t-1} \xrightarrow{(s^t j^t, -d_X^{t-1})} X^t$$

Since $(s^t \circ j^t, -d_X^{t-1})\binom{z}{x} = (s^t \circ j^t)(z) - d_X^{t-1}(x) = 0$, there exists $y' \in L^{t-1}$ such that $v^{t-1}(y') = x$. Hence v^{t-1} is an epimorphism.

Since A is noetherian, L^{t-1} is finitely generated, and hence, it admits a projective cover $p^{t-1} : P^{t-1} \to L^{t-1}$ with P^{t-1} being finitely generated. Set $d_P^{t-1} = j^t \circ u^{t-1} \circ p^{t-1}$ and $s^{t-1} = v^{t-1} \circ p^{t-1}$.

By induction, we have constructed finitely generated A-modules P^i, and A-linear maps $d_P^i : P^i \to P^{i-1}$ and also A-linear epimorphisms $s^i : P^i \to X^i$, for all $i \in \mathbb{Z}$, such that the conditions (1), (2), and (3) as stated above are satisfied for every $i \in \mathbb{Z}$. In particular, by Condition (2), we have $d_P^{i+1} \circ d_P^i = 0$; and by Conditions (2) and (3), we obtain

$$d_X^i s^i = d_X^i v^i p^i = s^{i+1} j^{i+1} u^i p^i = s^{i+1} d_P^i,$$

for all $i \in \mathbb{Z}$. That is, we have a bounded above complex $(P^\bullet, d_P^\bullet)$ of finitely generated projective A-modules and a morphism $s^\bullet : P^\bullet \to X^\bullet$ of complexes.

It remains to show that $H^i(s^\bullet) : H^i(P^\bullet) \to H^i(X^\bullet)$ is an isomorphism, for every $i \in \mathbb{Z}$. Indeed, let $p' \in \ker(d_P^i) = K^i$ be such that $s^i(p') \in \operatorname{im}(d_X^{i-1})$. That is, there exists $x' \in X^{i-1}$ such that $s^i(p') = d_X^{i-1}(x')$. Since v^{i-1} is an epimorphism, there exists $y' \in L^{i-1}$ such that $v^{i-1}(y') = x'$. This yields

$$(s^i j^i)(p') = s^i(p') = d_X^{i-1}(x') = d_X^{i-1} v^{i-1}(y') = s^i j^i u^{i-1}(y').$$

Thus, $(p' - u^{i-1}(y'), 0)^T$ lies in the kernel of $\left(s^i j^i, -d_X^{i-1}\right)$. By the exactness of the sequence

$$0 \longrightarrow L^{i-1} \xrightarrow{\binom{u^{i-1}}{v^{i-1}}} K^i \oplus X^{i-1} \xrightarrow{\left(s^i j^i, -d_X^{i-1}\right)} X^i,$$

we see that $p' - u^{i-1}(y') = u^{i-1}(z')$, for some $z' \in L^{i-1}$. That is, $p' = u^{i-1}(z' + y')$. Since p^{i-1} is an epimorphism, there exists $q \in P^{i-1}$ such that $z' + y' = p^{i-1}(q)$. Hence,

$$p' = u^{i-1}(z' + y') = (u^{i-1} p^{i-1})(q) = (j^i u^{i-1} p^{i-1})(q) = d_P^{i-1}(q) \in \operatorname{im}(d_P^{i-1}).$$

This shows that $H^i(s^\bullet)$ is an monomorphism.

Finally, let $x \in \ker(d_X^i)$. Then, $(0, x)^T$ is in the kernel of $(s^{i+1} j^{i+1}, -d_X^i)$. Using again the exactness of the sequence

$$0 \longrightarrow L^i \xrightarrow{\binom{u^i}{v^i}} K^{i+1} \oplus X^i \xrightarrow{\left(s^{i+1} j^{i+1}, -d_X^i\right)} X^{i+1},$$

we get $y \in L^i$ such that $v^i(y) = x$ and $u^i(y) = 0$. Since p^i is an epimorphism, $y = p^i(p)$ for some $p \in P^i$. Hence, $x = v^i p^i(p) = s^i(p)$ and $d_P^i(p) = j^{i+1} u^i p^i(p) = j^{i+1} u^i(y) = 0$. That is, $p \in \ker(d_P^i)$ such that $H^i(s^\bullet)(p + \operatorname{im}(d_P^{i-1})) = x + \operatorname{im}(d_X^{i-1})$. This shows that $H^i(s^\bullet)$ is an epimorphism. The proof of the theorem is completed.

A complex over Mod$A$ is called *perfect* if it is a bounded complex of finitely generated projective modules. The following result extends a well known result of Happel for the bounded derived category of finite dimensional modules over a finite dimensional algebra; see [14].

4.2.13 THEOREM. *Let A be a noetherian R-algebra with R being complete local noetherian, and let $Z^{\bullet}$ be a bounded complex over* mod$A$. *Then $D^b(\mathrm{Mod}A)$ has an Auslander-Reiten triangle ending in $Z^{\bullet}$ if and only if $Z^{\bullet}$ is isomorphic to an indecomposable perfect complex $P^{\bullet}$, and in this case, the Auslander-Reiten triangle is of the following form*

$$\nu P^{\bullet}[-1] \longrightarrow E^{\bullet} \longrightarrow P^{\bullet} \longrightarrow \nu P^{\bullet}.$$

Proof. By Theorem 4.2.12, there exists no loss of generality in assuming that $Z^{\bullet} = P^{\bullet}$ with $P^{\bullet} \in C^{-,b}(\mathcal{P}_A)$. Suppose that $D^b(\mathrm{Mod}A)$ has an Auslander-Reiten triangle

$$X^{\bullet} \xrightarrow{u^{\bullet}} Y^{\bullet} \xrightarrow{v^{\bullet}} P^{\bullet} \xrightarrow{\eta^{\bullet}} X^{\bullet}\,[1].$$

In particular, $\eta^{\bullet} \neq 0$. Write $W^{\bullet} = X^{\bullet}\,[1]$. We may assume that there exists some $n < 0$ such that $W^i = 0$ for all $i \notin [n, 0]$. By Lemma 4.2.10, there exists a morphism $w^{\bullet} : P^{\bullet} \to W^{\bullet} \in C^{-}(\mathrm{Mod}A)$ such that $\eta^{\bullet} = \tilde{w}^{\bullet}$. By Theorem 4.1.18, we have an exact triangle

$$\delta_{\geq n}(P^{\bullet}) \xrightarrow{\bar{\mu}^{\bullet}_n} P^{\bullet} \xrightarrow{\bar{\pi}^{\bullet}_n} \delta_{<n}(P^{\bullet}) \xrightarrow{\bar{\omega}^{\bullet}_n} \delta_{\geq n}(P^{\bullet})[1]$$

in $K^{-,b}(\mathcal{P}_A)$. Suppose that $\bar{\mu}^{\bullet}_n$ is not a retraction in $K^{-,b}(\mathcal{P}_A)$. By Lemma 4.2.10, $\tilde{\mu}^{\bullet}_n$ is not a retraction in $D^b(\mathrm{Mod}A)$. Hence, $\eta^{\bullet}\tilde{\mu}^{\bullet}_n = \tilde{w}\tilde{\mu}^{\bullet}_n = 0$, and hence, $\bar{w}^{\bullet}\bar{\mu}^{\bullet}_n = 0$. Therefore, there exist morphisms $h^i : \delta_{\geq n}(P^i) \to W^{i-1}$, $i \in \mathbb{Z}$, such that $w^i\mu^i_n = h^{i+1}d^i_P + d^{i+1}_W h^i$ for all $i \geq n$. Since $\mu^i_n = 1_{P^i}$ for $i \geq n$, we have $w^i = h^{i+1}d^i_P + d^{i+1}_W h^i$ for all $i \geq n$. If $i < n$, since $W^i = 0$, we also have $w^i = h^{i+1}d^i_P + d^{i+1}_W h^i$. That is, $\bar{w}^{\bullet} = 0$, and hence, $\eta^{\bullet} = 0$, a contradiction. This shows that $\bar{\mu}^{\bullet}_n$ is not a retraction in $K^{-,b}(\mathcal{P}_A)$. Thus, $\delta_{\geq n}(P^{\bullet}) \cong \delta_{<n}(P^{\bullet})[-1] \oplus P^{\bullet}$. Since $\delta_{\geq n}(P^{\bullet})$ lies in $K^b(\mathcal{P}_A)$, which is Krull-Schmidt by Proposition 4.2.9, it follows that $P^{\bullet}$ is isomorphic to a complex in $K^b(\mathcal{P}_A)$. Thus, $Z^{\bullet}$ is isomorphic to a perfect complex in $D^b(\mathrm{Mod}A)$.

Conversely, assume that $Z^\cdot = P^\bullet$ with $P^\cdot \in K^b(\mathcal{P}_A)$. By Proposition 4.2.9(2), $P^\bullet$ is strongly indecomposable in $K^b(\mathcal{P}_A)$, and hence, by Lemma 4.2.10, it is strongly indecomposable in $D^b(\mathrm{Mod}A)$. By Lemma 4.2.10, $\mathrm{End}_{K^b(\mathcal{P}_A)}(P^\bullet) \cong \mathrm{End}_{D^b(\mathrm{Mod}A)}(P^\bullet)$ and $\mathrm{End}_{K^b(\mathcal{I}_A)}(\nu P^\bullet) \cong \mathrm{End}_{D^b(\mathrm{Mod}A)}(\nu P^\bullet)$. Moreover, in view of Proposition 4.2.11,

$$\begin{aligned}\mathrm{End}_{D^b(\mathrm{Mod}A)}(\nu P^\bullet) &\cong D\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, \nu P^\bullet) \cong D^2\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, P^\bullet)\\ &\cong \mathrm{End}_{D^b(\mathrm{Mod}A)}(P^\bullet).\end{aligned}$$

Therefore, $\nu P^\bullet$ is also strongly indecomposable in $D^b(\mathrm{Mod}A)$. By Proposition 4.2.11, we have a functorial isomorphism

$$\varphi : D\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^\bullet, -) \to \mathrm{Hom}_{D^b(\mathrm{Mod}A)}(-, \nu P^\bullet).$$

By Theorem 2.1.16, $D^b(\mathrm{Mod}A)$ has an Auslander-Reiten triangle

$$\nu P^\bullet[-1] \longrightarrow E^\bullet \longrightarrow P^\bullet \longrightarrow \nu P^\bullet.$$

The proof of the theorem is completed.

4.3 Auslander-Reiten Triangles induced from Auslander-Reiten Sequences

Throughout this section, let A be a noetherian R-algebra with R being complete local noetherian. Let

$$\delta :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in $\mathrm{mod}A$. As described in Section 4.1, this sequence determines a morphism $\delta \in \mathrm{Hom}_{\mathrm{D^b(modA)}}(Z, X[1])$. Here we shall give another interpretation of this morphism. Let

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} Z \longrightarrow 0$$

be a projective resolution of Z. Then we can get the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "u_0"] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \\ & 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$

By $t^\bullet$ we denote the quasi-isomorphism

$$\begin{tikzcd} P^\bullet : & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\ & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots , \end{tikzcd}$$

and by $b^\bullet$ we denote the morphism

$$\begin{tikzcd} P^\bullet : & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots . \end{tikzcd}$$

This yields another morphism $\delta' \in \mathrm{Hom}_{D^b(\mathrm{mod}A)}(Z, X[1])$ represented by

$$\begin{tikzcd} & P^\bullet \arrow[dl, "\bar{t}^\bullet"'] \arrow[dr, "\bar{b}^\bullet"] & \\ Z & & X[1] \end{tikzcd}$$

4.3.1 Lemma. *Let* $\delta : 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be a short exact sequence in* mod$A$*. If* δ, δ' *are previously described morphisms in* $D^b(\mathrm{mod}A)$*, then* $\delta = \delta'$*.*

Proof. Recall that δ is represented by

$$\begin{tikzcd} & C_f^\bullet \arrow[dl, "\bar{s}^\bullet"'] \arrow[dr, "\bar{p}^\bullet"] & \\ Z & & X[1], \end{tikzcd}$$

where $C_f^{\bullet}$ is the mapping cone of f and $p^{\cdot}$ is the canonical projection, $s^{\cdot}$ is the quasi-isomorphism given by g. Denote by $e^{\cdot}$ the following morphism

$$\begin{tikzcd}
P^{\bullet}: & \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d, "u_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
C_f^{\bullet}: & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Obviously, $t^{\cdot} = s^{\cdot}e^{\cdot}$ and $b^{\cdot} = p^{\cdot}e^{\cdot}$. This yields a commutative diagram

$$\begin{tikzcd}
 & P^{\bullet} \arrow[dl, "t^{\bullet}"'] \arrow[dd, "e^{\bullet}"] \arrow[dr, "b^{\bullet}"] & \\
Z & & X[1] \\
 & C_f^{\bullet} \arrow[ul, "s^{\bullet}"] \arrow[ur, "p^{\bullet}"'] &
\end{tikzcd}$$

As a consequence, $\delta = \delta'$. The proof of the lemma is completed.

If Z is an indecomposable non-projective module in noe(A), then it admits a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow Z \longrightarrow 0$$

in noe(A). Applying the functor $\mathrm{Hom}_A(-, A)$, we obtain an exact sequence

$$\mathrm{Hom}_A(P_0, A) \xrightarrow{\mathrm{Hom}(f,A)} \mathrm{Hom}_A(P_1, A) \longrightarrow \mathrm{Tr}Z \longrightarrow 0$$

in noe(A^{op}). Applying the duality D, we obtain another exact sequence

$$0 \longrightarrow \mathrm{DTr}Z \longrightarrow \nu(P_1) \xrightarrow{\nu(f)} \nu(P_0)$$

in art(A), which is a minimal injective co-presentation of DTrZ. The following result is due to Auslander; see [4, (6.3)].

4.3.2 THEOREM ([4]). *Let A be a noetherian R-algebra with R being complete local noetherian. If $Z \in \mathrm{noe}(A)$ is indecomposable and non-projective, then there exists an Auslander-Reiten sequence*

$$0 \longrightarrow \mathrm{DTr}Z \longrightarrow E \longrightarrow Z \longrightarrow 0$$

in Mod$A$ *with* $\mathrm{DTr}Z \in \mathrm{art}(A)$.

4.3.3 LEMMA. *Let* $\delta :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be an Auslander-Reiten sequence in* Mod$A$ *with* $Z \in \mathrm{noe}(A)$. *If δ induces an Auslander-Reiten triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1]$$

in $D^b(\mathrm{Mod}A)$, *then* $\mathrm{pdim}(Z) \leq 1$ *and* $\mathrm{idim}(X) \leq 1$.

Proof. By Theorem 4.2.13, there exists an isomorphism $\theta^\bullet : P^\bullet \to Z$ where $P^\bullet$ is a bounded complex as follows,

$$P^\bullet :\ \cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0 \longrightarrow \cdots,$$

with $P_i \in \mathcal{P}_A$ for $i \geq 0$. By Proposition 4.2.10, $\theta^\bullet = \tilde{f}^\bullet$, for some morphism $f^\bullet : P^\bullet \to Z$. By Lemma 4.1.14, we know that f is a quasi-isomorphism. Therefore, it is easy to see that

$$(*)\quad \cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{f^0} Z \longrightarrow 0$$

is a projective resolution of Z in mod$A$, which we may assume to be a minimal projective resolution. By Theorem 4.2.13, there is an Auslander-Reiten triangle

$$\nu P^\bullet[-1] \longrightarrow E^\bullet \longrightarrow P^\bullet \longrightarrow \nu P^\bullet .$$

By the uniqueness of Auslander-Reiten triangle, we can get a commutative diagram

$$\begin{tikzcd}
\nu P^\bullet[-1] \arrow[r] \arrow[d] & E^\bullet \arrow[r] \arrow[d] & P^\bullet \arrow[r] \arrow[d] & \nu P^\bullet \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "\sigma"] & X[1],
\end{tikzcd}$$

where the vertical morphisms are isomorphisms. In particular, $X[1] \cong \nu P^\bullet$ in $D^b(\mathrm{Mod}A)$. Using an argument dual to the previous one, we obtain a quasi-isomorphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \nu P_2 \arrow[r, "\nu(d_2)"] & \nu P_1 \arrow[r, "\nu(d_1)"] & \nu P_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

in $C^b(\mathrm{Mod}A)$, where X is the component of degree -1. Hence we have

$$X \cong H^{-1}(\nu P^\bullet) = \ker(\nu(d_1))/\mathrm{im}(\nu(d_2)).$$

Moreover, since $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is an Auslander-Reiten sequence in $\mathrm{Mod}A$, by Theorem 4.3.2, we know $X = \mathrm{DTr}Z = \ker(\nu(d_1))$. Therefore, $\nu(d_2) = 0$. By Proposition 4.2.7, $d_2 = 0$. Thus the projective dimension of Z is less than or equal to 1. Moreover, we have a short exact sequence

$$0 \longrightarrow X \longrightarrow \nu P_0 \longrightarrow \nu P_1 \longrightarrow 0,$$

which means that injective dimension of X is less than or equal to 1. The proof of the lemma is completed.

4.3.4 THEOREM. *Let A be a noetherian R-algebra with R being complete local noetherian commutative, and let* $\mathrm{Mod}A$ *have an Auslander-Reiten sequence*

$$\delta:\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

with $Z \in \mathrm{noe}(A)$. Then $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\sigma} X[1]$ is an Auslander-Reiten triangle in $D^b(\mathrm{Mod}A)$ if and only if $\mathrm{pdim}(Z) \le 1$ and $\mathrm{idim}(X) \le 1$.

Proof. We need only to show the sufficiency. Assume that $\mathrm{pdim}(Z) \le 1$. Then there exists a minimal projective resolution of Z

$$\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} Z \longrightarrow 0.$$

By $P^\bullet$ we denote the complex

$$\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0 \longrightarrow \cdots .$$

As a consequence, we obtain a quasi-isomorphism $c^{\bullet}: P^{\bullet} \to Z$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

By Theorem 4.2.13, $D^b(\mathrm{Mod}A)$ has an Auslander-Reiten triangle

$$\nu P^{\bullet}[-1] \longrightarrow E^{\bullet} \longrightarrow P^{\bullet} \xrightarrow{\gamma^{\bullet}} \nu P^{\bullet}.$$

Moreover, we can complete the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "u_0"] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0.
\end{tikzcd}$$

By $b^{\bullet}$ we denote the morphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

By Lemma 4.3.1, we know that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} X[1]$ is an exact triangle in $D^b(\mathrm{Mod}A)$, where $\delta = \bar{b}^{\bullet}/\bar{c}^{\bullet}$. Since $\delta \neq 0$, we have $\bar{b}^{\bullet} \neq 0$.

Since $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an Auslander-Reiten sequence, $X \cong \mathrm{DTr}Z$. That is, $0 \longrightarrow X \xrightarrow{s} \nu P_1 \xrightarrow{\nu(d_1)} \nu P_0 \longrightarrow 0$ is a short exact sequence, which yields a quasi-isomorphism $e^{\bullet}: X[1] \to \nu P^{\bullet}$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "s"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \nu P_1 \arrow[r, "\nu(d_1)"] & \nu P_0 \arrow[r] & \cdots
\end{tikzcd}$$

Denote $h^{\bullet} = e^{\bullet} b^{\bullet} : P^{\bullet} \to \nu P^{\bullet}$. Now we claim that $\tilde{h}^{\bullet} : P^{\bullet} \to \nu P^{\bullet}$ is in the $\mathrm{End}_{D^b(\mathrm{Mod}A)}(P^{\bullet})$-socle of $\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(P^{\bullet}, \nu P^{\bullet})$. In view of Proposition 4.2.10, we see that this claim is equivalent to $\bar{h}^{\bullet} : P^{\bullet} \to \nu P^{\bullet}$ being in the

$\mathrm{End}_{K^b(\mathrm{Mod}A)}(P^\bullet)$-socle of $\mathrm{Hom}_{K^b(\mathrm{Mod}A)}(P^\bullet, \nu P^\bullet)$. Let $\bar{w}^\bullet : P^\bullet \to P^\bullet$ be a non-retraction morphism in $K^b(\mathrm{Mod}A)$ as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "w_1"] & P_0 \arrow[r] \arrow[d, "w_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

This induces a commutative diagram with exact rows in $\mathrm{Mod}A$

$$\begin{tikzcd}
0 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "w_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "w_0"] & Z \arrow[r] \arrow[d, "w_2"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "u_0"] & Z \arrow[r] \arrow[d, "1_Z"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "1_X"] & Y \arrow[r, "g"] \arrow[d, "v_1"] & Z \arrow[r] \arrow[d, "v_0"] & 0 \\
0 \arrow[r] & X \arrow[r, "s"] & \nu P_1 \arrow[r, "\nu(d_1)"] & \nu P_0 \arrow[r] & 0.
\end{tikzcd}$$

Sine $\bar{w}$ is not a retraction, neither is w_2. Hence, there exists a morphism $\alpha : Z \to P_0$ such that $w_2 = g\alpha$. Write $k = u_0 w_0 - \alpha\varepsilon$. Then $gk = gu_0 w_0 - g\alpha\varepsilon = gu_0 w_0 - w_2\varepsilon = 0$. We get a morphism $\beta : P_0 \to X$ such that $f\beta = k$. We know $fu_1 w_1 = u_0 w_0 d_1 = (\alpha\varepsilon + f\beta)d_1 = f\beta d_1$. Since f is a monomorphism, $u_1 w_1 = \beta d_1$. Now consider $h^\bullet w^\bullet = e^\bullet d^\bullet w^\bullet$,

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "w_1"] & P_0 \arrow[r] \arrow[d, "w_2"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "u_1"] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "s"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \nu P_1 \arrow[r, "\nu(d_1)"] & \nu P_0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

We know

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "su_1w_1"'] & P_0 \arrow[r] \arrow[d, "0"] \arrow[dl, "s\beta"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \nu P_1 \arrow[r, "\nu(d_1)"] & \nu P_0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

commutes, that is $\bar{h}^{\bullet}\bar{w}^{\bullet} = 0$. This establishes our claim. Moreover, since $e^{\bullet}$ is a quasi-isomorphism and $\bar{b} \neq 0$, we know $\bar{h} \neq 0$. By Corollary 2.1.7, we have $\tilde{h}^{\bullet} = \gamma^{\bullet}\theta^{\bullet}$ for some automorphism of $P^{\bullet}$. As a consequence, we have an Auslander-Reiten triangle

$$\nu P^{\bullet}[-1] \longrightarrow E^{\bullet} \longrightarrow P^{\bullet} \xrightarrow{\tilde{h}^{\bullet}} \nu P^{\bullet}$$

in $D^b(\mathrm{Mod}A)$. Furthermore, we establish a commutative diagram in $D^b(\mathrm{Mod}A)$.

$$\begin{tikzcd}
\nu P^{\bullet}[-1] \arrow[r] \arrow[d, "\tilde{e}^{\bullet -1}[-1]"] & E^{\bullet} \arrow[r] \arrow[d] & P^{\bullet} \arrow[r, "\tilde{h}^{\bullet}"] \arrow[d, "\tilde{c}^{\bullet}"] & \nu P^{\bullet} \arrow[d, "\tilde{e}^{\bullet -1}"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "\sigma"] & X[1],
\end{tikzcd}$$

where the vertical arrows are isomorphisms. Therefore, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\sigma} X[1]$ is an Auslander-Reiten triangle in $D^b(\mathrm{mod}A)$. The proof of the theorem is completed.

Conclusion

In this dissertation, by using non-degenerate bilinear forms, we give some new description of Auslander-Reiten triangles which lead to some existence theorems of Auslander-Reiten triangles in extension-closed subcategory of triangulated categories. Applying the existence theorem to the bounded derived category of all modules of a noetherian algebra over a complete noetherian local commutative ring, we establish a sufficient and necessary condition to have an Auslander-Reiten triangle. There is still a lot to do along this direction. For example, we could continue to work on the following problems.

1. The notion of an Auslander-Reiten sequence is introduced to a general additive category; see [21]. Up to now, there exists no existence theorem of Auslander-Reiten sequences in an additive category. We notice that a weakly abelian category is an additive category in which each morphism has a pseudo kernel and a pseudo cokernel. On the other hand, abelian categories and triangulated categories are weakly abelian categories. Therefore, it would be interesting to find an existence theorem of almost sequence in a weakly abelian category, which unifies the existence theorems for abelain categories and triangulated categories.

2. Let A be a noetherian R-algebra with R being complete local noetherian commutative. If $D^b(\mathrm{Mod}A)$ has an Auslander-Reiten triangle

$$X^{\bullet} \longrightarrow Y^{\bullet} \longrightarrow Z^{\bullet} \longrightarrow X^{\bullet}[1] ,$$

we conjecture that Z is necessarily in $D^b(\mathrm{mod}A)$, and hence, a perfect complex by Theorem 4.2.13.

Bibliography

[1] L. Alonso, A. Jeremías, and M. J. Souto, "Constructions of t-structures and equivalences of derived categories", Trans. Amer. Math. Soc. 355 (2003) 2523-2543.

[2] I. Assem, "Algèbres et Modules: Cours et Exercices," Enseignement des Mathématiques (Les Presses de l'Université d'Ottawa-MASSON, Ottawa-Paris, 1997).

[3] I. Assem, A. Beligiannis, and N. Marmaridis, "Right triangulated categories with right semi-equivalences", Proc. Geiranger 1996, CMS Conf. Proc. Vol. 24 (1998) 17-37.

[4] M. Auslander, "Functors and morphisms determined by objects", Lecture Notes in Pure Appl. Math. 37 (Dekker, New York, 1978) 1 - 244.

[5] M. Auslander and I. Reiten, "Representation theory of artin algebras III", Comm. Alebra 3 (1975) 239 - 294.

[6] M. Auslander and I. Reiten, "Representation theory of artin algebras IV", Comm. Alebra 5 (1977) 443 - 518.

[7] M. Auslander, I. Reiten, and S. O. SmalØ, "Representation Theory of Artin Algebras", Cambrige Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[8] R. Bautista and S. Liu "Covering theory for linear categories with application to derived categories," J. Algebra 406 (2014) 173-225.

[9] R. Bautista, S. Liu, and C.Paquette, "Representation theory of strongly locally finite quivers", Proc. London Math. Soc. 106 (2013) 97 - 162.

[10] A. A. Beĭlinson, J. Beinstein, and P. Deligne, " Faisceaux pervers", Analysis and topology on singular spaces I (Luminy, 1981), 5 - 171, Astérisque 100 (Soc. Math. France, Paris, 1982).

[11] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, "Algebraic D-Modules", Perspectives in Mathematics 2 (Academic Press, Boston, MA, 1987).

[12] W. Bruns and J. Herzog, "Cohen-Macaulay rings", Cambridge Studies in Advanced Mathematics 39 (Cambridge University Press, Cambridge, 1993).

[13] D. Happel, "Triangulated Categories in the Representation Theory of Finite Dimensional Algebras", London Mathematical Society Leture Notes Series 119 (Cambridge University Press, Cambridge, 1988).

[14] D. Happel, "Auslander-Reiten triangles in derived categories of finite-dimensional algebras", Proc. Amer Math. Soc. 112 (1991) 3.

[15] D. Happel, "On the derived category of a finite-dimensional algebra", Comment. Math. Helv. 62 (1987) 339 - 389.

[16] P. Jørgensen, "Aulander-Reiten Triangles in Subcategories", J. K-Theory 3 (2009) 583 - 601.

[17] H. Krause, "A short proof for Auslander's defect formula", Linear Algebra Appl. 365 (2003) 267 - 270.

[18] H. Krause, "Auslander-Reiten Theory via Brown Representability", K-Theory 20 (2000) 331 - 344.

[19] H. Krause and J. Le, " The Auslander-Reiten formula for complexes of modules", Adv. Math. 207 (2006) 133 - 148.

[20] J. Le and X. Chen, "Karoubianness of a triangulated category", J. Algebra 310 (2007) 452 - 457.

[21] S. Liu, "Auslander-Reiten theory in a Krull-Schmidt category", Sao Paulo J. Math. Sci. 4 (2010) 425 - 472.

[22] S. Liu, P. Ng, and C. Paquette, "Almost split sequence and approximation", Algebr. Represent. Ther. 16 (2013) 1809 - 1827.

[23] D. Milicic, "Lectures on Derived Categories", preprint, available at www.math.utah.edu/ milicic/Eprints/dercat.pdf

[24] I. Reiten and M. Van Den Bergh, "Noetherian hereditary abelian categories satisfying Serre duality", J. Amer. Math. Soc. 15 (2002) 295 - 366.

[25] J. J. Rotman, "An introduction to homological algebra", Pure and Applied Mathematics, 85 (Academic Press, New York-London, 1979).

[26] J. - L. Verdier, "Catégories dérivées", Leture notes in Math. Vol. 569, Springer-Verlag. (1977).

[27] C. A. Weibel, "An introduction to homological algebra", Cambridge studies in advanced mathematics 38 (Cambridge University Press, Cambridge, 1994).