Auslander-Reiten components with bounded short cycles

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joint with

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Motivation and objective

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5. $\text{rad}^n(\text{mod} A)$, the $n$-th power of $\text{rad}(\text{mod} A)$. 

Note: $\text{rad}^\infty(\text{mod} A) := \bigcap_{n \geq 0} \text{rad}^n(\text{mod} A)$, infinite radical.
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6. $\text{rad}^\infty(\text{mod} A) := \cap_{n \geq 0} \text{rad}^n(\text{mod} A)$, infinite radical.
7. Let $\Gamma_A$ be the AR-quiver of $A$, with AR-translation $\tau$. 
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\textbf{Remark}

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Proposition (Igusa-Todorov)

If $X_0 \xrightarrow{f_1} X_1 \xrightarrow{} \cdots \xrightarrow{} X_{n-1} \xrightarrow{f_n} X_n$ is a sectional path of irreducible maps in $\text{ind} A$, then $dp(f_n \cdots f_1) = n$. 
A *cycle* of length $n$ in $\text{mod} A$ is a sequence of non-zero non-isomorphisms in $\text{ind} A$. If $n = 2$, then $\sigma$ is called a *short cycle*. The depth of $\sigma$ is defined by $\text{dp}(\sigma) = \max \{ \text{dp}(f_1), \ldots, \text{dp}(f_n) \}$. If all the $X_i$ belong to a subquiver $\Gamma$ of $\Gamma_A$, then $\sigma$ is called a *cycle in $\text{add}(\Gamma)$*. 
A *cycle* of length $n$ in $\text{mod}A$ is a sequence

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Cycles

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Theorem

An artin algebra $A$ is representation-finite if

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2. ind $A$ contains no short cycle (Happel-Liu).
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If $A$ is representation-finite, then $\text{rad}^\infty(\text{mod } A) = 0$, and consequently, $\Gamma_A$ is short-cycle-bounded.
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Examples of short-cycle-bounded subquivers

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3. Reiten and Skowronski introduced the notion of generalized double tilted algebra.

Theorem

An artin algebra $A$ is generalized double tilted $\iff \Gamma_A$ has a faithful, generalized standard and short-cycle-bounded component.
Cuts

Definition

A connected full subquiver $\Delta$ of $\Gamma_A$ is called $\tau$-rigid if $\text{Hom}_A(X, \tau Y) = 0$ for all $X, Y \in \Delta$. A cut is provided, for arrow $X \to Y$ in $\Gamma_A$, that if $X \in \Delta$, then $Y$ or $\tau Y$, not both, belongs to $\Delta$; if $Y \in \Delta$, then $X$ or $\tau^{-1}X$, not both, belongs to $\Delta$. 
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Characterizations of tilted algebras

Theorem (Liu)

An artin algebra $A$ is a tilted algebra $\Leftrightarrow \Gamma_A$ contains a faithful $\tau$-rigid cut $\Delta$; and in this case, $\Delta$ is a slice.

Corollary (Liu)

If $\Delta$ is a $\tau$-rigid cut of $\Gamma_A$, then the quotient algebra $B = A/{\text{ann}}(\Delta)$ is tilted with $\Delta$ being a slice of $\Gamma_B$. 
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If $\Delta$ is a $\tau$-rigid cut of $\Gamma_A$, then the quotient algebra $B = A/\text{ann}(\Delta)$ is tilted with $\Delta$ being a slice of $\Gamma_B$. 
Let $C$ be a connected component of $\Gamma_A$. 
Semi-stable components

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4. A left or right stable component of $\mathcal{C}$ is called \emph{semi-stable component} of $\Gamma_A$. 

The core of $\mathcal{C}$ is the full subquiver generated by the modules lying on $P \Rightarrow I$, with $P$ projective and $I$ injective.
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5. The *core* of $\mathcal{C}$ is the full subquiver generated by the modules lying on $P \rightsquigarrow I$, with $P$ projective and $I$ injective.
Proposition

Let $\Gamma$ be an infinite semi-stable component of $\Gamma_A$. 

Thus, $\text{add}(\Gamma)$ has short cycles of arbitrarily large depths.

If $\Gamma$ contains no oriented cycle, it contains cuts of $\Gamma_A$; and if such a cut is not $\tau$-rigid, then $\text{add}(\Gamma)$ contains short cycles of infinite depth.
Proposition

Let $\Gamma$ be an infinite semi-stable component of $\Gamma_A$.

1. If $\Gamma$ has oriented cycles, then it has infinite sectional paths.

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Thus, $\text{add}(\Gamma)$ has short cycles of arbitrarily large depths.

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Theorem

Let $C$ be short-cycle-bounded connected component of $\Gamma_A$. 

Main result

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Let $C$ be short-cycle-bounded connected component of $\Gamma_A$. Then $C$ consists of

- a finite core containing all possible oriented cycles,
- some infinite left stable components $\Gamma_1, \ldots, \Gamma_r$ with $r \geq 0$, 

where each $\Gamma_i$ has $\tau$-rigid cut $\Delta_i$ such that $B_i = A / \text{ann}(\Delta_i)$ is tilted and all the predecessors of $\Delta_i$ in $C$ belong to the connecting component of $\Gamma_B$, and

each $\Theta_i$ has $\tau$-rigid cut $\Sigma_i$ such that $C_i = A / \text{ann}(\Sigma_i)$ is tilted and all the successors of $\Delta_i$ in $C$ belong to the connecting component of $\Gamma_C$. 
Theorem

Let $ \mathcal{C} $ be short-cycle-bounded connected component of $ \Gamma_A $. Then $ \mathcal{C} $ consists of

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- some infinite right stable components $ \Theta_1, \ldots, \Theta_s $ with $ s \geq 0 $. 

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Let $A = kQ/I$ be radical squared zero, where

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We have a short-cycle-bounded AR-component as follows:
The algebra $A$ is of representation-finite if and only if there exists a bound for the depths of short cycles in $\text{ind} A$. 

**Theorem**

$\Gamma_A$ has at most finitely many short-cycle-bounded components; and each of them has only finitely many $\tau$-orbits.
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