The derived AR-components of algebras with radical squared zero

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Advance in Representation Theory of Algebras VI

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Question

What are the shapes of the AR-components of $D^b(\mathrm{mod}\,A)$?

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Objective

In case $\mathrm{rad}^2(A) = 0$, describe the AR-components of $D^b(\mathrm{mod}\, A)$.

Our methodology

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2. Representation theory of infinite quivers.

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Proposition

The G-orbit category $\mathcal{A}/G$ is Hom-finite Krull-Schmidt k-category with a canonical embedding

$$\sigma : \mathcal{A} \to \mathcal{A}/G : X \mapsto X; f \mapsto f.$$

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$\exists$ commutative diagram

$$\begin{array}{ccc} \mathcal{A} & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\pi} & \\ \mathcal{A}/G & \xrightarrow{\sim} & \mathcal{B}. \end{array}$$

AR-quiver under Galois covering

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1. The functor $\pi : \mathcal{A} \to \mathcal{B}$ induces a Galois G-covering of translation quivers $\pi : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{B}}$.
2. The connected components of $\Gamma_{\mathcal{B}}$ are the images

$$\pi(\Gamma),$$

where Γ ranges over the connected components of $\Gamma_{\mathcal{A}}$.

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   - *perfect* if $X^\cdot \cong$ bounded complex over $\mathrm{proj}\, A$.

Grading period of Q

Given walk $w = \alpha_1^{e_1} \cdots \alpha_r^{e_r}$ in Q, $\alpha_i \in Q_1$, $e_1 = \pm 1$, write

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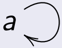

$$\cdots \to (a, -2) \to (a, -1) \to (a, 0) \to (a, 1) \to (a, 2) \to \cdots$$

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2. It has an automorphism

$$\rho : \tilde{Q} \longrightarrow \tilde{Q} : (a, n) \mapsto (a, n + r_Q).$$

3. Setting $G =< \rho >$ yields a Galois G-covering of quivers:

$$\pi : \tilde{Q} \longrightarrow Q : (a, n) \mapsto a.$$

Representations of $\tilde{Q}^{\mathrm{op}}$

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3. A representation M of $\tilde{Q}^{\mathrm{op}}$ is *finitely co-presented* if $\exists$

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Remark

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2. AR-components of $D^b(\mathrm{rep}^-(\tilde{Q}^{\rm op}))$ have been described by Bautista, Liu and Paquette.

Group action on $D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}}))$

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3. The group
   $$\mathfrak{G} = < \vartheta >$$
   acts admissibly on $D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}}))$.

Derived Koszul push-down functor

Theorem

There exists Galois $\mathfrak{G}$-covering

$$\mathfrak{F}_\pi : D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})) \longrightarrow D^b(\mathrm{mod} A).$$

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   - perfect $\Leftrightarrow M^\cdot \cong \mathfrak{F}_\pi(M)$ for some $M \in \mathrm{rep}^b(\tilde{Q}^{\mathrm{op}})$.

Translation quiver with a section

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Definition

A connected full subquiver Δ of Γ is called a *section* if it is

1. acyclic ;
2. convex in Γ; and
3. meets every τ-orbit exactly once.

Proposition

If Γ contains a section Δ, then it embeds in $\mathbb{Z}\Delta$.

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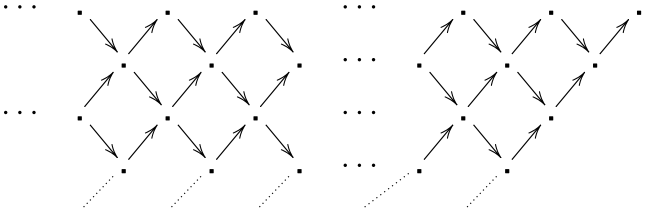
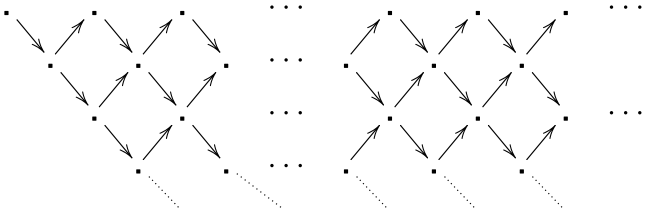
1. The simple complexes in $\mathscr{C}$ form a section $\cong \tilde{Q}$.
2. As a consequence, $\mathscr{C}$ embeds in $\mathbb{Z}\tilde{Q}$.
3. The components $\mathscr{C}[i]$, $i \in \mathbb{Z}/r_Q\mathbb{Z}$, are the components of $\Gamma_{D^b(\operatorname{mod} A)}$ containing simple complexes.

Translation quiver of shape $\mathbb{N}^{-}\mathbb{A}_{\infty}^{-}$



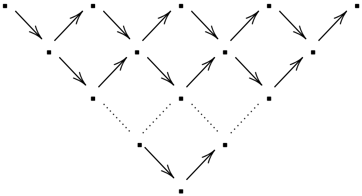
It contains a right-most section $\cong \mathbb{A}_{\infty}^{-}$.

Translation quivers of shape $\mathbb{N}^+\mathbb{A}_\infty^+$



It contains a left-most section $\cong \mathbb{A}_\infty^+$.

Wings



It contains a left-most section and a right-most section $\cong \mathbb{A}_n$.

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   - $\mathbb{Z}\mathbb{A}_\infty/{<}\tau^n{>}$; only if Q is Euclidean with $r_Q = 0$.

AR-components without simple complexes

Theorem

Let $\mathscr{C}$ component of $\Gamma_{D^b(\mathrm{mod}\, A)}$ without simple complexes.

1. If $\mathscr{C}$ contains only perfect complexes, then it is of shape
   - $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}^-\mathbb{A}_\infty^-$ or
   - $\mathbb{Z}\mathbb{A}_\infty/<\tau^n>$; only if Q is Euclidean with $r_Q = 0$.
2. Otherwise, $\mathscr{C}$ is a wing or of shape $\mathbb{N}\mathbb{A}_\infty^+$, and whose non-perfect complexes generate the left-most section.

Theorem

If $\mathrm{gdim}(A) < \infty$, then AR-components of $D^b(\mathrm{mod} A)$ are of shapes

$$\mathbb{Z}\tilde{Q},\ \mathbb{Z}\mathbb{A}_\infty,\ \mathbb{Z}\mathbb{A}_\infty / < \tau^n > .$$

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   (1) n components being sectional double infinite path;
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4. In other cases, $\Gamma_{D^b(\mathrm{mod}\, A)}$ has infinitely many components.

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   (1) $\mathcal{R} \cong \mathbb{Z}\mathbb{A}_\infty$, of perfect complexes;

   (2) $\mathcal{L}$ is a sectional double infinite path

$$\cdots \longrightarrow S[-2] \longrightarrow T[-1] \longrightarrow S[0] \longrightarrow T[1] \longrightarrow S[2] \longrightarrow \cdots$$