Tilted String Algebras

François Huard and Shiping Liu

Introduction

Tilted algebras, that is endomorphism algebras of tilting modules over a hereditary algebra, have been one of the main objects of study in representation theory of algebras since their introduction by Happel and Ringel [**10**]. As a generalization, Happel, Reiten and Smalø studied endomorphism algebras of tilting objects of a hereditary abelian category which they call quasi-tilted algebras [**9**]. The latter has attracted a lot of attention of recent investigations. So far all complete characterizations of tilted or quasi-tilted algebras are module-theoretical [**9, 10**]. On the other hand, Gabriel's theorem says that a finite-dimensional algebra over an algebraically closed field is determined, up to Morita equivalence, by its bound quiver [**6**] . It is then natural and interesting to characterize tilted or quasi-tilted algebras in terms of their bound quiver. This has been done for tilted algebras of type $\mathbf{A}_n$, $\tilde{\mathbf{A}}_n$ and for tame concealed algebras [**1, 11, 15**]. As the problem in general seems very difficult, if not impossible, we shall consider it for string algebras, that is monomial biserial algebras [**3, 5**]. As results we shall find some simple combinatorial criteria for a string algebra to be tilted or quasi-tilted. As a consequence, this will enable one to construct a lot of new examples of tilted algebras. Finally we shall determine all quasi-tilted string algebras which are not tilted.

1. Preliminaries

We first fix some terminology and notations which will be used throughtout this paper. Let Q be a finite quiver. For an arrow α of Q, denote by $s(\alpha)$ its start-point, by $e(\alpha)$ its end-point and by α^{-1} its formal inverse with start-point $s(\alpha^{-1}) = e(\alpha)$ and end-point $e(\alpha^{-1}) = s(\alpha)$, and write $(\alpha^{-1})^{-1} = \alpha$.

A *walk* in Q of length $n(> 0)$ is a sequence $w = c_1 \cdots c_n$ with c_i an arrow or the inverse of an arrow such that $e(c_{i+1}) = s(c_i)$ for $1 \le i < n$. We call the c_i *edges* of w, in particular c_1 the *initial edge* and c_n the *terminal edge*. Moreover, we define $s(w) = s(c_1)$ and $e(w) = e(c_n)$ and say that w is a walk from $s(w)$ to $e(w)$. Finally we define $w^{-1} = c_n^{-1} \cdots c_1^{-1}$. A *trivial walk* at a vertex a is the trivial path ε_a with $e(\varepsilon_a) = s(\varepsilon_a)$.

A walk w in Q is called *reduced* if w is trivial or $w = c_1 \cdots c_n$ such that $c_{i+1} \neq c_i^{-1}$ for all $1 \le i < n$. A non-trivial reduced walk $w = c_1 \cdots c_n$ is called a *reduced cycle* if $s(w) = e(w)$ and $c_n \neq c_1^{-1}$; and a *simple cycle* if in addition $s(c_1), \ldots, s(c_n)$ are distinct. Note that a reduced cycle can be written in many equivalent forms by choosing different vertex as its start-point.

Let $w = c_1 \cdots c_n$ be a non-trivial reduced walk in Q. Let $w_1 = c_i \cdots c_j$ with $1 \le i \le j \le n$ and $w_2 = c_r \cdots c_t$ with $1 \le r \le t \le n$ be subwalks of w. We say that w_1, w_2 *point to the same direction* in w if there are paths p, q of Q such that either $w_1 = p,\ w_2 = q$ or $w_1 = p^{-1},\ w_2 = q^{-1}$ and otherwise they *point to opposite directions* in w.

Let k be an algebraically closed field. Denote by kQ^+ the ideal of the path algebra kQ generated by the arrows of Q. If I is an ideal of kQ such that $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$ for some $n \ge 2$, then the pair (Q, I) is called a *bound quiver*. We say that a bound quiver (Q', I') is a *full bound subquiver* of (Q, I) if Q' is a full subquiver of Q and $I' = kQ' \cap I$.

Let (Q, I) be a bound quiver. A path p in Q is called a *zero-path* if $p \in I$. A zero-path is called a *zero-relation* on Q if none of its proper subpaths is a zero-path. Let $w = c_1 \cdots c_n$ be a non-trivial reduced walk in Q. We say that a subwalk $u = c_i \cdots c_{i+r}$ is a zero-relation contained in w if $u = p$ or p^{-1} with p a zero-relation on Q. By saying that a reduced cycle contains no zero-relation we mean none of its written forms contains a zero-relation. Note that a zero-relation on Q may appear many times in a reduced walk.

Let A be a finite-dimensional basic k-algebra. Then $A \cong kQ/I$ with (Q, I) a bound quiver. We shall identify the category of the finite-dimensional right A-modules with that of the finite-dimensional representations of (Q, I).

1.1. Definition [3]. *A k-algebra A is called a* string algebra *if $A \cong kQ/I$ with (Q, I) a bound quiver satisfying the following:*

(1) *I is generated by a set of paths.*

(2) *each vertex of Q is start-point or end-point of at most two arrows.*

(3) *for an arrow α, there is at most one arrow β such that $\alpha\beta \notin I$ and at most one arrow γ such that $\gamma\alpha \notin I$.*

In the sequel by saying that $A = kQ/I$ is a string algebra, we mean that (Q, I) is a bound quiver satisfying the above-stated conditions. We now state some known facts about string algebras.

1.2. Proposition [3, 13]. *Let $A = kQ/I$ be a string algebra. Then*

(1) *A is of tame representation type.*

(2) *A is of finite representation type if and only if all reduced cycle of Q contains at least one zero-relation.*

(3) *A is of directed representation type if and only if all reduced cycle of Q contains at least two zero-relations pointing to opposite directions.*

The first two statements follow directly from the facts that each indecomposable module over a string algebra is either a string module or a band module (see sections 2 and 3 for definitions) and that there is at most finitely many isoclasses of string modules of each dimension. The third one is a reformulation of a result of de la Peña [**13**].

2. Quasi-tilted string algebras

In this section we shall find a simple combinatorial criterion for deciding whether a string algebra is quasi-tilted or not. Recall that a finite-dimensional k-algebra is quasi-tilted if and only if its global dimension is at most two and each indecomposable module is either of projective dimension at most one or of injective dimension at most one [**9**].

Let $A = kQ/I$ be a string algebra. A reduced walk in Q is called a *string* if it contains no zero-relation. One says that a string w *starts* or *ends in a deep* if there is no arrow γ such that $\gamma^{-1}w$ or $w\gamma$ is a string, respectively; and it *starts* or *ends on a peak* if there is no arrow δ such that δw or $w\delta^{-1}$ is a string, respectively.

If $w = \varepsilon_a$ is the trivial path at a, then the string module $M(w)$ is the simple module at a. Let now $w = c_1 c_2 \cdots c_n$ be a non-trivial string. For

$0 \leq i \leq n$, let $U_i = k$; and for $1 \leq i \leq n$, let U_{c_i} be the identity map sending $x \in U_i$ to $x \in U_{i+1}$ if c_i is an arrow and otherwise the identity map sending $x \in U_{i+1}$ to $x \in U_i$. The string module $M(w)$ is then defined as follows: for a vertex a, $M(w)_a$ is the direct sum of the spaces U_i such that $s(c_i) = a$ if a appears in w, otherwise $M(w)_a = 0$; for an arrow α, $M(w)_\alpha$ is the direct sum of the maps U_{c_i} such that $c_i = \alpha$ or $c_i^{-1} = \alpha$ if α appears in w, otherwise $M(w)_\alpha$ is the zero map.

For a vertex a of Q, we denote by $P(a)$ and $I(a)$ the indecomposable projective and injective module at a, respectively. It is then well-known that $P(a) = M(u^{-1}v)$, where u, v are paths starting with a such that $u^{-1}v$ is a string starting and ending in a deep; and $I(a) = M(pq^{-1})$, where p, q are paths ending with a such that pq^{-1} is a string starting and ending on a peak.

2.1. Lemma. *Let $A = kQ/I$ be a string algebra. Let $w = p_1^{-1}q_1 \cdots p_r^{-1}q_r$ be a string, where the p_i, q_j are paths which are non-trivial for $1 < i \leq r$ and $1 \leq j < r$. If the projective dimension of $M(w)$ is greater than one, then one of the following holds:*

(1) *there is a non-trivial path z_i with $2 \leq i \leq r$ such that $p_i z_i$ and $q_{i-1} z_i$ are both zero-paths.*

(2) *there is a non-trivial path z_1 such that $z_1^{-1}w$ is a reduced walk and $p_1 z_1$ is a zero-path while $p_1 \alpha$ is not a zero-path, where α is the initial arrow of z_1.*

(3) *there is a non-trivial path z_{r+1} such that wz_r is a reduced walk and $q_r z_{r+1}$ is a zero-path while $q_r \beta$ is not a zero-path, where β is the initial arrow of z_{r+1}.*

Proof. Assume that the projective dimension of $M(w)$ is greater than one. For each $1 \leq i \leq r$, write $a_i = s(q_i)$ and let u_i, v_i be the paths of non-negative length such that $u_i^{-1}p_i^{-1}q_i v_i$ is a string which starts and ends in a deep. Then $P(a_i) = M(u_i^{-1}p_i^{-1}q_i v_i)$. It is easy to see that $P = \oplus_{i=1}^r P(a_i)$ is the projective cover of $M(w)$. Let K be the kernel of the canonical epimorphism from P to $M(w)$. By calculating the dimensions, we see that $K \cong \oplus_{i=1}^{r+1} K_i$, where $K_1 = 0$ if u_1 is trivial and otherwise $K_1 = M(u^{-1})$ with u the path so that $u_1 = \alpha u$ for an arrow α; $K_i = M(u_i^{-1} v_{i-1})$ for $2 \leq i \leq r$; and $K_{r+1} = 0$ if v_r is trivial and otherwise $K_{r+1} = M(v)$ with v the path so that $v_r = \delta v$ for an arrow δ. Since $M(w)$ is of projective dimension greater than one, at least one of the K_i is not projective.

Suppose first that $K_i = M(u_i^{-1}v_{i-1})$ is not projective for some $2 \le i \le r$. Then $u_i^{-1}v_{i-1}$ does not start or not end in a deep. In the first case, there is an arrow β_i such that $\beta_i^{-1}u_i^{-1}v_{i-1}$ is a string. In particular the initial arrow of $u_i\beta_i$ is not contained in the path v_{i-1}. Hence $q_{i-1}u_i\beta_i$ is a zero-path. Moreover $p_iu_i\beta_i$ is a zero-path since $u_i^{-1}p_i^{-1}q_iv_i$ is a string starting in a deep. Let $z_i = u_i\beta_i$ in this case. Similarly in the second case there is an arrow γ_i such that $q_{i-1}v_i\gamma_i$ and $p_iv_i\gamma_i$ are zero-paths. Let $z_i = v_i\gamma_i$ in this case. Hence (1) holds.

Suppose now that K_1 is not projective. Then $u_1 = \alpha u$ with α an arrow and $K_1 = M(u^{-1})$. Since $M(u^{-1})$ is not projective, u^{-1} does not start or not end in a deep. In the first case, there is an arrow β_1 such that $\beta_1^{-1}u^{-1}$ is a string. However $p_1\alpha u\beta_1 = p_1u_1\beta_1$ is a zero-path since $u_1^{-1}p_1^{-1}q_1v_1$ is a string starting in a deep. Let $z_1 = \alpha u\beta_1$ in this case. Otherwise there is an arrow γ_1 such that $u^{-1}\gamma_1$ is a string. Note then that u is non-trivial. Therefore $\alpha\gamma_1$ is a zero-relation. Let $z_1 = \alpha\gamma_1$ in this case. Thus (2) holds. Similarly we can show that (3) holds if K_{r+1} is not projective. The proof is completed.

The following notion is essential for our characterization of quasi-tilted string algebras.

2.2. Definition. *Let $A = kQ/I$ be a string algebra. A reduced walk w is called a* sequential pair of zero-relations *in (Q, I) if w contains exactly two zero-relations and these two zero-relations point to the same direction in w.*

Note that the two zero-relations in a sequential pair of zero-relations can be the same zero-relation on the quiver. For instance one can get such a sequential pair of zero-relations from a simple cycle containing exactly one zero-relation.

2.3. Lemma. *Let $A = kQ/I$ be a string algebra such that there is no sequential pair of zero-relations in (Q, I). Then each string module is either of projective dimension at most one or of injective dimension at most one.*

Proof. Assume that there is a string w such that $M(w)$ has both projective dimension and injective dimension greater than one. Let $w = p_1^{-1}q_1 \cdots p_r^{-1}q_r$, where the p_i, q_j are paths which are non-trivial for $1 < i \le r,\ 1 \le j < r$. We shall obtain a sequential pair of zero-relations by considering only the case where p_1 is non-trivial and q_r is trivial, since the other cases can be treated similarly. Assume that this is the case. Then by Lemma 2.1, there is a path

$z_1 = \alpha z_1'$ with α an arrow such that $p_1 z_1$ is a zero-path whereas $p_1\alpha$ is not; or there is a non-trivial path z_i with $2 \le i \le r$ such that $p_i z_i$ and $q_{i-1} z_i$ are zero-paths; or there is a zero-path $z_{r+1} = \beta z_{r+1}'$ with β an arrow so that $w z_r$ is a reduced walk.

We now write $w = q_0 p_1^{-1} q_1 \cdots q_{r-1} p_r^{-1}$ with q_0 a trivial path. Then by the dual of Lemma 2.1, there is a zero-path $y_0 = y_0' \gamma$ with γ an arrow such that $y_0 w$ is a reduced walk; or there is a non-trivial path y_i for some $1 \le i \le r-1$ such that both $y_i q_i$ and $y_i p_i$ are zero-paths; or there is a path $y_r = y_r' \delta$ with δ an arrow so that $y_r p_r$ is a zero-path whereas δp_r is not.

Suppose first that $y_0 = y_0' \gamma$ exists. If $z_1 = \alpha z_1'$ exists, then $\gamma\alpha$ is a zero-relation since $p_1\alpha$ is not a zero-path. Therefore $y_0 \alpha = y_0' \gamma \delta$ is a sequential pair of zero-relations. If z_j exists for some $1 < j < r$, then $y_0 p_1^{-1} q_1 \cdots p_{j-1}^{-1} q_{j-1} z_j$ is a sequential pair of zero-relations.

Suppose now that y_i exists for some $0 < i < r$. If z_j exists for some $1 \le j \le i$, then $y_i p_i q_{i-1}^{-1} \cdots q_j^{-1} p_j z_j$ is sequential pair of zero-relations. If z_j exists for some $i < j < r$, then $y_i q_i p_{i+1}^{-1} \cdots p_{j-1}^{-1} q_{j-1} z_j$ is a sequential pair of zero-relations.

Suppose finally that $y_r = y_r' \delta$ exists. If z_j exists for some $1 \le j \le r$, then $y_r p_r q_{r-1}^{-1} \cdots q_j^{-1} p_j z_j$ is a sequential pair of zero-relations. If $z_{r+1} = \beta z_{r+1}'$ exists, then $\delta\beta$ is a zero-relation since $p_r \delta$ is non-zero. Hence $y_r' \delta \beta$ is a sequential pair of zero-relations. This completes the proof of the lemma.

Let $A = kQ/I$ be a string algebra. A reduced cycle $w = c_1 c_2 \cdots c_n$ in Q is called a *band* if w is not a power of a reduced cycle of less length and all its powers contain no zero-relation. Let ϕ be an indecomposable automorphism of a k-vector space V. For $1 \le i \le n$, define $V(i) = V$. For $1 \le i \le n-1$, let f_{c_i} be the identity map from $V(i)$ to $V(i+1)$ if c_i is an arrow; and otherwise the identity map from $V(i+1)$ to $V(i)$, and let f_{c_n} be the map sending $x \in V(n)$ to $\phi(x) \in V(1)$ if c_n is an arrow; and otherwise the map sending $x \in V(1)$ to $\phi^{-1}(x) \in V(n)$. The band module $N = N(w, \phi)$ determined by w and ϕ is then defined as follows: for each vertex a of Q, if a appears in w, then N_a is the direct sum of the spaces $V(i)$ such that $s(c_i) = a$, and otherwise N_a is the zero-space. For each arrow α of Q, if α appears in w, then N_α is the direct sum of the maps f_{c_i} such that $c_i = \alpha$ or $c_i = \alpha^{-1}$; and otherwise N_α is the zero-map. For $1 \le i \le n$, denote by h_i the canonical projection from $N_{s(c_i)}$ to $V(i)$. Then for $1 \le i \le n$, $h_i f_{c_i} = N_{c_i} h_{i+1}$ if c_i is an arrow; and $N_{c_i^{-1}} h_i = h_{i+1} f_{c_i}$ if c_i is the inverse of an arrow (we identify $i+1$ with its remainder divided by n).

2.4. Lemma. *Let $A = kQ/I$ be a string algebra such that there is no sequential pair of zero-relations in (Q, I). Then each band module is either of projective dimension at most one or of injective dimension at most one.*

Proof. Let $w = c_1c_2 \cdots c_n$ be a band. Let $N = N(w, \phi)$ be a band module as defined above, and we keep all the notations. Assume that the injective and projective dimension of N are both greater than one. We shall find a sequential pair of zero-relations. Note that $\mathrm{DTr}(N) = N$ [**3**, section 3]. Thus $\mathrm{Hom}(D(_AA), N) \neq 0$ and $\mathrm{Hom}_A(N, A) \neq 0$ [**13**, (2.4)]. Let a_0 be a vertex such that there is a non-zero homomorphism f from $I(a_0)$ to N. Note that $I(a_0) = M(pq^{-1})$, where p, q are paths so that pq^{-1} is a string starting and ending in a peak. Clearly $I(a_0)$ is not simple since B is indecomposable. Thus f factors through the socle factor of $I(a_0)$. Therefore we may assume that $p = u\alpha_0$ with α_0 an arrow such that there is a non-zero homomorphism g from $M(u)$ to N. Let $u = \alpha_{t-1} \cdots \alpha_1$ with $\alpha_i : a_{i+1} \to a_i$ an arrow for $0 < i < t$ (when $t = 1$, $u = \varepsilon_{a_1}$). Note that the homomorphism g from $M(u)$ to N consists of a family of linear maps $g_a : M(u)_a \to N_a$, where a runs over the vertices of Q. Let r with $1 \leq r \leq t$ be minimal so that g_{a_r} is non-zero. We shall show that a_r appears in w as a sink. In fact, there is some $1 \leq m \leq n$ such that $g_{a_r}h_m \neq 0$. Hence $a_r = s(c_m)$. Assume that c_m is an arrow, say from a_r to b. Then $g_b = 0$, this follows from the minimality of r if $b = a_{r-1}$ and otherwise from the fact that $M(u)_b = 0$. We now have $g_{a_r}h_m f_{c_m} = M(u)_{c_m} g_b h_{m+1} = 0$ (we identify $m+1$ with its remainder divided by n). This is contrary to the fact that f_{c_m} is an isomorphism. Thus c_m is the inverse of an arrow. Using the same argument we see that c_{m-1} is an arrow. Therefore a_r does appear in w as a sink.

Note that the band w is not an oriented cycle. Thus up to equivalence, we can write $w = p_1q_1^{-1} \cdots p_sq_s^{-1}$, where the p_i, q_i are non-trivial paths with $s(p_1) = s(q_s)$. Now $a_r = e(p_{s_0}) = e(q_{s_0})$ for some $1 \leq s_0 \leq s$. We want to show that both $q_{s_0}\alpha_{r-1} \cdots \alpha_0$ and $p_{s_0}\alpha_{r-1} \cdots \alpha_0$ are zero-paths. It suffices to show that both q_{s_0} and p_{s_0} contain a vertex which does not appear in $\alpha_{t-1} \cdots \alpha_r$. Suppose on the contrary that this is not the case. Then p_{s_0} or q_{s_0} is a subpath of $\alpha_{t-1} \cdots \alpha_r$. Assume that $q_{s_0} = \alpha_{r+d} \cdots \alpha_r$ with $0 \leq d < t - r$. Since $a_r = s(c_m)$, we have $c_{m+i} = \alpha_{r+i}^{-1}$ for $0 \leq i \leq d$. Note that $g_{a_{r+1}}h_{m+1}f_{c_m} = M(u)_{\alpha_r} g_{a_r} h_m \neq 0$. Hence $g_{a_{r+1}} \neq 0$. Inductively $g_{a_{r+d+1}} \neq 0$. Now $a_{r+d+1} = s(q_{s_0}) = s(p_{s_0+1})$ (we identify s_0+1 with its remainder divided by s). Hence c_{m+d+1} is an arrow (we identify $m + d + 1$ with it remainder divided by n), say from a_{r+d+1} to x. Then

$$g_{a_{r+d+1}} h_{m+d+1} f_{c_{m+d+1}} = M(u)_{c_{m+d+1}} g_x h_{m+d+2}.$$

However $g_x = 0$ since x does not appear in u, this is contrary to the fact that $f_{c_{m+d+1}}$ is an isomorphism. Thus $q_{s_0}\alpha_{r-1}\cdots\alpha_0$ and $p_{s_0}\alpha_{r-1}\cdots\alpha_0$ are zero-paths. Using the fact that $\mathrm{Hom}_A(N, A) \neq 0$, we can dually show that there is a non-trivial path q so that for some $1 \leq t_0 \leq s$, both qq_{t_0} and qp_{t_0} are zero-paths. It is now easy to see that we have a sequential pair of zero-relations. The proof of the Lemma is completed.

We are now ready to get our main result of this section.

2.5. Theorem. *Let $A = kQ/I$ be a string algebra. Then A is quasi-tilted if and only if there is no sequential pair of zero-relations in (Q, I).*

Proof. Assume first that (Q, I) contains no sequential pair of zero-relations. In particular there is no path in Q containing two overlapping zero-relations. Therefore the global dimension of A is at most two [**7**, (1.2)]. Let M be an indecomposable A-module. Then M is either a string module or a band module [**3**]. Applying Lemmas 2.3 and 2.4, we see that either the projective dimension or the injective dimension of M is at most one. Thus A is quasi-tilted.

Conversely let q be a sequential pair of zero-relations of (Q, I). If q is a path containing two overlapping zero-relations, then the global dimension of A is greater than two [**7**, (1.2)]. Hence A is not quasi-tilted. Otherwise we may assume that q is of the form $q = z_1 w z_2$, where z_1, z_2 are two paths which are zero-relations and $w = p_1^{-1}q_1 \cdots p_r^{-1}q_r$ is a string such that the p_i, q_j are paths which are non-trivial for $1 < i \leq r$, $1 \leq j < r$. We shall prove that $M(w)$ has projective and injective dimensions both greater than one. First write $z_2 = \delta u \rho$ with δ, ρ arrows and u a path of non-negative length. For each $1 \leq i \leq r$, let $a_i = s(p_i)$ and let u_i, v_i be the paths such that $u_i^{-1}p_i^{-1}q_i v_i$ is a string starting and ending in a deep. Note that $u_i^{-1}v_{i-1}$ is a string for all $1 < i \leq r$ and $v_r = \delta u$. Moreover $P(a_i) = M(u_i^{-1}p_i^{-1}q_i v_i)$, and $P = \oplus_{i=1}^{r} P(a_i)$ is the projective cover of $M(w)$. Let K be the kernel of the canonical epimorphism from P to $M(w)$. Then $K \cong \oplus_{i=1}^{r+1} K_i$, where $K_1 = 0$ if u_1 is trivial and otherwise $K_1 = M(u^{-1})$ with u the path so that $u_1 = \alpha u$ for an arrow α; $K_i = M(u_i^{-1}v_{i-1})$ for $2 \leq i \leq r$; and $K_{r+1} = M(v)$. Since $v\rho$ is not a zero-path, K_{r+1} is not projective. This implies that $M(w)$ has projective dimension greater than one. Dually one can show that the injective dimension of M is greater than one. Therefore A is not quasi-tilted. This completes the proof of the theorem.

3. Tilted string algebras

In this final section we shall find a sufficient and necessary condition for a string algebra to be tilted. Moreover we shall determine all quasi-tilted string algebras which are not tilted.

Let $A = kQ/I$ be a string algebra, and let Θ be a simple cycle of Q containing no zero-relation. Let α be an arrow of Q. We say that α *enters* Θ if $e(\alpha) \in \Theta$ whereas $s(\alpha) \notin \Theta$. Similarly we say that α *leaves* Θ if $s(\alpha) \in \Theta$ whereas $e(\alpha) \notin \Theta$. Finally we say that α is *attached* to Θ if it enters or leaves Θ. Moreover, we call an arrow β a *left* or *right annihilator* of α if $\beta\alpha$ or $\alpha\beta$ is a zero-relation, respectively. It follows easily from the definition of a string algebra that α has a left or right annihilator in Θ if α leaves or enters Θ, respectively.

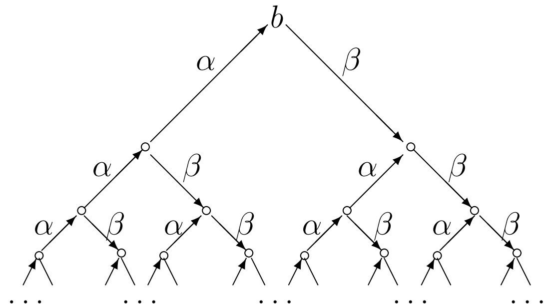
3.1. Lemma. *Let $A = kQ/I$ be a connected quasi-tilted string algebra. Let Θ be a simple cycle of Q containing no zero-relation. Let α be an arrow entering Θ and contained in only one zero-relation on Q. If w is a reduced walk having α as its terminal edge, then w contains no zero-relation and the start-point of each edge of w is not on any reduced cycle of Q.*

Proof. By Theorem 2.5, there is no sequential pair of zero-relations in (Q, I). By assumption, α has exactly one right annihilator β in Θ. Write $\Theta = \beta u$ with u a reduced walk such that $e(u) = s(\beta) = e(\alpha)$. Let $w = c_n \cdots c_1 c_0$ be a reduced walk, where $c_0 = \alpha$ and $c_i = \alpha_i$ or α_i^{-1} with α_i an arrow for $1 \le i \le n$. Assume that w contains a zero-relation, say $c_s \cdots c_{s_0}$ with $s_0 \ge 0$ minimal. If $c_s \cdots c_{s_0}$ is a path, then $c_s \cdots c_{s_0} \cdots c_0 \beta$ is a path containing two zero-relations, which is impossible. If $c_s \cdots c_{s_0}$ is the inverse of a path, then we consider the reduced walk $w_1 = \alpha\beta u c_0^{-1} \cdots c_{s_0}^{-1} \cdots c_s^{-1}$. Note that $\beta u c_0^{-1} \cdots c_{s_0}^{-1} \cdots c_{s-1}^{-1}$ contains no zero-relation by the minimality of s_0 and the hypotheses on Θ and α. Thus w_1 is sequential pair of zero-relations, which is a contradiction.

To prove the second part of the statement, we first show that $s(c_i)$ is not on Θ for all $0 \le i \le n$. If this is not the case, let t with $0 \le t \le n$ be minimal such that $s(c_t) \in \Theta$. Then $t \ge 1$ and $e(c_t) \notin \Theta$. Thus α_t has a left or right annihilator γ in Θ. Let $c = \gamma$ or γ^{-1} such that cc_t is a walk. Then cc_t is a zero-relation contained in the reduced walk $cc_t \cdots c_1c_0$, which is impossibe as we have shown.

Suppose now that there is some minimal r with $0 \le r \le n$ such that $s(c_r)$ is on a reduced cycle Θ_0. If c_r does not belong to Θ_0, then c_r is attached to Θ_1, and hence it has a left or right annihilator δ in Θ_0. Let $d = \delta$ or δ^{-1} such that dc_r is a walk. Then dc_r is a zero-relation contained in the reduced walk $dc_r \cdots c_0$, which is impossible by (1). If c_r belongs to Θ_0, then $r = 0$ by the minimality of r. Thus we can write $\Theta_0 = d_1 \cdots d_m c_0$, where d_i or d_i^{-1} is an arrow for $1 \le i \le m$ and $s(d_1) = e(c_0) = e(\alpha) \in \Theta$, which is contrary to our previous claim. The proof is completed.

Recall that a *branch* with pivot b is a finite connected full bound subquiver containing the vertex b of the following infinite bound quiver whose zero-relations are all possible $\alpha\beta$:



Let (Γ, J) be a bound quiver, and let B be a branch with pivot b and underlying quiver Δ. One says that a bound quiver (Q, I) is obtained from (Γ, J) by adding B at b if $Q = \Gamma \cup \Delta$, $\Gamma \cap \Delta = \{b\}$, and all relation on Q has its support either in Γ or in Δ [**14**, (4.4)].

3.2. Lemma. *Let $A = kQ/I$, Θ and α be as in Lemma 3.1. Let (Q', I') be the full bound subquiver of (Q, I) so that the vertices of Q' are those of Q and the arrows of Q' are those of Q different from α. Let B be the connected component of (Q', I') containing $s(\alpha)$ and C the one containing $e(\alpha)$. Then*

(1) *Q' is the disjoint union of B and C.*

(2) *B is a branch with pivot b, where $b = s(\alpha)$.*

(3) *Let (Γ, J) be the full bound subquiver of (Q, I) generated by C and α. Then (Q, I) is obtained from (Γ, J) by adding the branch B at b.*

Proof. By Lemm 3.1, B and C are disconnected in (Q', I'). Thus Q' is the disjoint union of B and C since Q is connected. Let $\mathit{\Delta}$ be the underlying quiver of B. Then $\mathit{\Delta} \cap \mathit{\Gamma} = \{b\}$ and $\mathit{\Delta} \cup \mathit{\Gamma} = Q$. Moreover α appears in any reduced walk w with $s(w) \in \mathit{\Delta}$ and $e(w) \in \mathit{\Gamma}$. Therefore all zero-relation on Q lies completely either in $\mathit{\Gamma}$ or in $\mathit{\Delta}$ since α is contained in only one zero-relation $\alpha\gamma$, where γ is the right annihilator of α in $\mathit{\Theta}$.

It remains to show that B is a branch with pivot b. First note that B is a tree by Lemma 3.1. Thus for each arrow δ of B, there is a unique reduced walk $w(\delta)$ in B from $s(\delta)$ to b. We define δ to be positive if the initial edge of $w(\delta)$ is δ; and to be negative otherwise. One can easily conclude that B is a branch with pivot b from the following properties of B.

(a) *There is at most one arrow δ_+ in B starting with b and at most one δ_- ending with b. Moreover if δ_+, δ_- both exist, then $\delta_+\delta_-$ is a zero-relation.* In fact $s(\alpha) = b$ implies that there is at most one arrow in B starting with b since $\alpha \notin B$. If there are two arrows δ_1, δ_2 ending with b, then either $\delta_1\alpha$ or $\delta_2\alpha$ is a zero-relation, which contradicts Lemma 3.1. Suppose now that δ_+, δ_- are arrows with $e(\delta_+) = s(\delta_-) = b$. Note that $\delta_+\alpha$ is not a zero-relation by Lemma 3.1. Hence $\delta_+\delta_-$ is a zero-relation since A is a string algebra.

(b) *Let a be a vertex of B other than b. Then there is in B at most one arrow starting with a of each sign and at most one ending with a of each sign. Moreover there are at most three arrows starting or ending with a in B.* In fact, if δ_1, δ_2 are two distinct positive arrows starting with a, then $w(\delta_1), w(\delta_2)$ are two distinct reduced walks from a to b, which is impossible. If δ_1, δ_2 are two distinct negative arrows starting with a, then $w(\delta_1) = w(\delta_2) = \gamma^{-1}v$, where γ is an arrow ending with a and v is a reduced walk with $e(v) = b$. We may then assume that $\gamma\delta_1$ is a zero-relation. Therefore $\delta_1^{-1}\gamma^{-1}v\alpha$ is a reduced walk containing a zero-relation, which is contrary to Lemma 3.1. Thus there is at most one negative arrow of B starting with a. Similarly one can show that there is at most one arrow ending with a of each sign. Suppose now that there are four arrows $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ starting or ending with a. We may assume that $s(\gamma_1) = s(\gamma_2) = a = e(\gamma_3) = e(\gamma_4)$, and further γ_1 is positive. Then $w(\gamma_1) = \gamma_1 v_1$, where v_1 is a reduced walk with $e(v_1) = b$. By the definition of a string algebra, we can assume that $\gamma_3\gamma_1$ is a zero-relation. Hence $\gamma_3\gamma_1 v_1\alpha$ is a reduced walk in Q containing a zero-relation, which is contrary to Lemma 3.1.

(c) *A path $p = \delta_1 \cdots \delta_n$ in B with δ_1 negative or δ_n positive is not a zero-path.* In fact, assume that p is a zero-path. If δ_n is positive, then $w(\delta_n) = \delta_n v_2$, where v_2 is a reduced walk with $e(v_2) = b$. Hence $\delta_1 \cdots \delta_n v_1\alpha$

is a reduced walk containing a zero-relation, which is impossible by Lemma 3.1. If δ_1 is negative, then $\delta_n^{-1} \cdots \delta_1^{-1} w(\delta_1)\alpha$ is a reduced walk containing a zero-relation, which is also contrary to Lemma 3.1.

(d) *If $\delta_+\delta_-$ is a path with δ_+ positive and δ_- negative, then it is zero-relation. Moreover all zero-relation in B is of this form.* In fact, suppose that $e(\delta_+) = s(\delta_-) = a$. We may assume to $a \neq b$ by (1). Since δ_- is negative, $w(\delta_-) = du$, where $d = \gamma$ or γ^{-1} with γ an arrow different from δ_- and u is a reduced walk with $e(u) = b$. If $s(\gamma) = a$, then γ is positive. So $\delta_+\gamma$ is not a zero-relation by (c). Hence $\delta_+\delta_-$ is a zero-relation. If $e(\gamma) = a$, then γ is negative by (b). Hence $\gamma\delta_-$ is not a zero-relation by (c). Thus $\delta_+\delta_-$ is a zero-relation. Finally let $p = \delta_1 \cdots \delta_n$ with $n \geq 2$ be a zero-relation in B. Then δ_1 is positive and δ_n is negative by (c). Thus there is some $1 \leq i \leq n$ such that δ_i is positive and δ_{i+1} is negative. Hence $\delta_i\delta_{i+1}$ is a zero-relation. Hence $i = 1$ and $p = \delta_1\delta_2$. This completes the proof of the Lemma.

3.3. Lemma. *Let $A = kQ/I$ be a quasi-tilted string algebra, and let Θ be a simple cycle in Q containing no zero-relation. If there is an arrow entering Θ and one leaving Θ, then*

(1) *all arrow attached to Θ is contained in only one zero-relation on Q,*

(2) *the arrows attached to Θ are pairwise disjoint, and*

(3) *the right annihilator of an arrow entering Θ and the left annihilator of an arrow leaving Θ point to opposite directions in Θ.*

Proof. Let α be an arrow entering Θ and β be one leaving Θ. Let γ be a right annihilator of α and δ a left annihilator of β in Θ. Assume that γ, δ point to the same direction in Θ. Then Θ contains a reduced walk u_1 with initial edge γ and terminal edge δ. Hence $\alpha u_1 \beta$ is a sequential pair of zero-relations, which is a contradiction to Theorem 2.5. Thus (3) holds.

To show (1), let $p = \alpha_1 \cdots \alpha_n$ be a zero-relation on Q with $\alpha_r = \alpha$ for some $1 \leq r \leq n$. Then $r < n$ since otherwise $\alpha_1 \cdots \alpha_n \gamma$ would be a sequential pair of zero-relations. If $\alpha_{r+1} \cdots \alpha_n$ does not lie in Θ, then there is a minimal t with $r < t \leq n$ such that α_t is not in Θ. Note then that α_t leaves Θ, and hence it has a left annihilator γ_t in Θ. If $t = r+1$, then $e(\gamma_{r+1}) = s(\alpha_{r+1}) = e(\alpha_r) = s(\gamma)$. Thus γ_{r+1}, γ point to the same direction in Θ, which is contrary to what we have proved. If $t > r+1$, then $\alpha_{r+1} \cdots \alpha_{t-1}$ lies in Θ. Since p is a zero-relation, $\alpha_{r+1} \neq \gamma$ and $\alpha_{t-1} \neq \gamma_t$. Then γ, γ_t point to the same direction in Θ, which is again contrary to what we have proved. Thus $\alpha_{r+1} \cdots \alpha_n$ lies in Θ. Hence we can write $\Theta = \alpha_{r+1} \cdots \alpha_n u_2 \gamma^{-1}$ with u_2 a reduced walk. If $\alpha_{r+1} \neq \gamma$, then α_{r+1}, δ point to the same direction in Θ.

Thus Θ contains a reduced walk u_3 with initial edge α_{r+1} and terminal edge δ. This implies that $\alpha_1 \cdots \alpha_{r+1} \cdots \alpha_n u_2 \gamma^{-1} u_3 \beta$ is a sequential pair of zero-relations, which is a contradiction. Hence $\alpha_{r+1} = \gamma$. Consequently $p = \alpha\gamma$. We can dually show that $\delta\beta$ is the only zero-relation containing β.

It remains to show (2). If $s(\beta) = e(\alpha)$, then δ, γ point to the same direction in Θ, which is impossible. If $e(\beta) = s(\alpha)$, then $\delta\beta\alpha\gamma$ is a sequential pair of zero-relations. Therefore α, β are disjoint. Let now $\alpha' \neq \alpha$ be another arrow entering Θ. By Lemma 3.1, $s(\alpha') \neq s(\alpha)$. Assume that $e(\alpha') = e(\alpha)$. Since A is a string algebra, Θ contains two arrows γ, γ' starting with $e(\alpha)$. Since $\alpha\gamma$ is the only zero-relation containing α, γ' is the right annihilator of α' in Θ. This implies that γ', δ point to the same direction in Θ, which is contrary to (3). Therefore α, α' are disjoint. Similarly if $\beta' \neq \beta$ is another arrow leaving Θ, then β, β' are disjoint. The proof is completed.

We are now ready to have our promised criterion for deciding a string algebra is tilted or not.

3.4. Theorem. *Let $A = kQ/I$ be a string algebra. Then A is tilted if and only if the following conditions are satisfied:*

(1) *there is no sequential pair of zero-relations in (Q, I);*

(2) *if Θ is a simple cycle of Q containing no zero-relation, then the arrows attached to Θ either all enter Θ or all leave Θ.*

Proof. Assume first that A is tilted. Then A is quasi-tilted. Hence (1) is satisfied by Theorem 2.5. Suppose now that there are in Θ a simple cycle Θ containing no zero-relation, an arrow α entering Θ and an arrow β leaving Θ. By Lemma 3.3, α, β are disjoint. Moreover α is contained in only one zero-relation $\alpha\gamma$ and β is contained in only one zero-relation $\beta\delta$, where γ, δ are arrows in Θ. Let now (Q', I') be the full bound subquiver of (Q, I) generated by Θ, α and β. Combining Lemmas 3.3 and 3.1, we infer that (Q', I') is convex in (Q, I). Thus $A' = kQ'/I'$ is tilted since A is [**8**, (6.5)]. Let $\Gamma_{A'}$ be the Auslander-Reiten quiver of A'. It is easy to see that the indecomposable projective A'-module at $s(\alpha)$ is in a ray tube and the others are in a preprojective component of $\Gamma_{A'}$. Dually the indecomposable injective A'-module at $e(\beta)$ is in a coray tube and the others are in a preinjective component of $\Gamma_{A'}$. As a consequence the complete slice would lie in a regular component of Γ', which is well-known to be impossible since A' is tame. Hence (2) is also satisfied.

Conversely suppose that (Q,I) satisfies both (1) and (2). We may further assume that (Q,I) is connected. By Theorem 2.5, A is quasi-tilted. If all possible simple cycle in Q contains zero-relations, then A is of finite representation type, and hence tilted [**9**, (3.6)]. Assume now that there is a simple cycle Θ which contains no zero-relation. If there is no arrow attached to Θ, then A is the hereditary algebra $k\Theta$. Otherwise let $\alpha_1, \cdots, \alpha_t$ be the arrows attached to Θ, which we may assume all enter Θ. Let γ_i be a right annihilator of α_i in Θ for $1 \le i \le t$.

We first consider the case where there is some α_i, say α_1 is contained in two distinct zero-relations. One of these is $\alpha_1\gamma_1$, and let the other one be $p = \beta_1 \cdots \beta_m$ with $\alpha_1 = \beta_r$ for some $1 \le r \le m$. Then $r < m$ by (1). It follows from (2) that $\beta_{r+1} \cdots \beta_m$ lies completely in Θ. Let $a = s(p)$ and $b = e(p)$. Write $p = \beta_1 u \beta_m$ with u a path containing no zero-relation. Note that β_{r+1} is different from γ_1 since otherwise $p = \alpha_1\gamma_1$. Thus the string module $M(u)$ is a direct summand of the radical of $P(a)$. Moreover it is easy to see that $M(u)$ is also a direct summand of the socle factor of $I(b)$. Thus $P(a)$ and $I(b)$ lie in the same connected component of the Auslander-Reiten quiver of A. Hence A is tilted since A is quasi-tilted [**4**, (5.3)].

It remains to consider the case where each α_i with $1 \le i \le t$ is contained in only one zero-relation, that is $\alpha_i\gamma_i$. Then the γ_i are distinct since A is a string algebra. Let $b_i = s(\alpha_i)$ and $a_i = e(\alpha_i)$ for $1 \le i \le t$. By Lemma 3.1, the b_i are distinct. Denote by (Q', I') the full bound subquiver of (Q,I) generated by Θ and the arrows $\alpha_1, \cdots, \alpha_t$. By Lemma 3.2, (Q,I) is obtained from (Q', I') by adding a branch at each vertex b_i. Note that $\mathrm{rad}P(b_i) = M(u_i)$, where u_i is the maximal subpath (maybe trivial) of Θ starting with a and not containing γ_i. Thus $\mathrm{rad}P(b_i)$ lies in the mouth of a non-homogeneous tube of the tame hereditary algebra $k\Theta$. Since the γ_i are distinct, the $\mathrm{rad}P(b_i)$ are pairwise non-isomorphic, and hence pairwise orthogonal. Therefore A is a domestic tubular extension of $k\Theta$, and hence tilted [**14**, (4.9)]. The theorem is now established.

We would like to point out that the characterization of tilted gentle algebras stated in [**12**] is not complete. In fact the statement there states essentially only the first condition of Theorem 3.4. Nevertheness it is true that all tilted gentle algebras are of type $\mathbf{A}_n$ or $\tilde{\mathbf{A}}_n$. However, by using the above result, it is easy to construct tilted string algebras of quite arbitrary types.

3.5. Definition. *A bound quiver is said to be of* type $\tilde{\mathbf{A}}_{n,r,t}$ *with* n, r, t *positive if*

(1) *the quiver consists of a non-oriented cycle* Θ *of type* $\tilde{\mathbf{A}}_n$ *and* r *arrows* $\alpha_1, \ldots, \alpha_r$ *entering* Θ *and* t *arrows* $\beta_1, \ldots, \beta_t$ *leaving* Θ *with the* $r+t$ *arrows* α_i, β_j *pairwise disjoint*;

(2) *the relations are* $\alpha_i\gamma_i$ *with* $1 \leq i \leq r$ *and* $\delta_j\beta_j$ *with* $1 \leq j \leq s$*, where the* γ_i, δ_j *are arrows in* Θ *such that each pair* γ_i, δ_j *point to opposite directions in* Θ*.*

We are now able to determine all quasi-tilted string algebras which are not tilted.

3.6. Theorem. *Let* $A = kQ/I$ *be a connected string algebra. Then* A *is a quasi-tilted algebra which is not tilted if and only if* (Q, I) *is obtained from a bound quiver of type* $\tilde{\mathbf{A}}_{n,r,t}$ *by adding a branch at each of the vertices not on the cycle. Moreover in this case,* A *is iterated tilted of type* $\tilde{\mathbf{A}}_m$*.*

Proof. Assume that $A = kQ/I$ is quasi-tilted and not tilted. By Theorems 2.5 and 3.4, there are in Q a simple cycle Θ containing no zero-relation, an arrow entering Θ and an arrow leaving Θ. Let $\alpha_1, \ldots, \alpha_r$ be the arrows entering Θ and $\beta_1, \ldots, \beta_s$ the ones leaving Θ. By Lemma 3.3., the $r+t$ arrows α_i, β_j are disjoint. Moreover each α_i is contained in exactly one zero-relation $\alpha_i\gamma_i$ and each β_j is contained in exactly one zero-relation $\beta_j\delta_j$, where γ_i, δ_j are arrows in Θ such that $\gamma_i,\ \delta_j$ point to opposite directions for all $1 \leq i \leq r, 1 \leq j \leq t$. Let (Q', I') be the full bound subquiver of (Q, I) generated by Θ and the arrows α_i, β_j with $1 \leq i \leq r; 1 \leq j \leq t$. Then (Q', I') is a bound quiver of type $\tilde{\mathbf{A}}_{n,r,t}$. Moreover by Lemma 3.2 and its dual, (Q, I) is obtained from (Q', I') by by adding a branch at each of $s(\alpha_1), \ldots, s(\alpha_r), e(\beta_1), \ldots, e(\beta_t)$.

Conversely let (Q', I') be a bound quiver of type $\tilde{\mathbf{A}}_{n,r,t}$ with $\Theta, \alpha_i, \gamma_i, \beta_j, \delta_j$ as defined in Definition 3.5. Assume that (Q, I) is obtained from (Q', I') by adding a branch D_i at $s(\alpha_i)$ for each $1 \leq i \leq r$ and a branch E_j at $e(\beta_j)$ for each $1 \leq j \leq t$. Clearly A is a string algebra. Hence A is not tilted by Theorem 3.4. Moreover (Q, I) satisfies all the conditions as stated in part (iv) of Theorem (A) of [**2**]. Thus A is iterated tilted of type $\tilde{\mathbf{A}}_m$. It remains to show that A is quasi-tilted. Assume that this is not the case. Then (Q, I) contains sequential pairs of zero-relations by Theorem 2.5. Note that all zero-relation on Q is of length two. Thus there is a reduced walk $w = c_1c_2 \cdots c_{s-1}c_s$ with $s \geq 3$, where c_1, c_2, c_3 and c_4 are arrows such that c_1c_2

and $c_{s-1}c_s$ are the only zero-relations contained in w. By definition each zero-relation is completely contained either in (Q', I') or in a branch. Note that each branch contains no sequential pair of zero-relations and that all reduced walk lying completely in a branch starting or ending with the pivot contains no zero-relation. Thus both c_1c_2 and $c_{s-1}c_s$ are in (Q', I'). We consider only the case $c_1c_2 = \alpha_{i_0}\gamma_{i_0}$ for some $1 \le i_0 \le r$. We now show that $c_2 \cdots c_{s-1}$ lies in Θ. In fact if this is not the case, let i_1 with $2 < i_1 \le s-1$ be minimal such that c_{i_1} is not in Θ, then either $c_{i_1} = \alpha_i^{-1}$ for some $1 \le i \le r$ or $c_{i_1} = \beta_j$ for some $1 \le j \le t$. Suppose that $c_{i_1} = \alpha_i^{-1}$. Then c_{i_1+1} is in the branch D_i since $c_{i_1+1} \ne \alpha_i$. As a consequence $c_{i_1+1} \cdots c_{s-1}c_s$ is reduced walk in D_i satring with the pivot $s(\alpha_i)$ et containing a zero-relation, which is impossible. Similarly it is impossible that $c_{i_1} = \beta_j^{-1}$ with $1 \le j \le t$. Therefore $c_2 \cdots c_{s-1}$ is contained in Θ. In particular $c_{s-1}c_s = \delta_{j_0}\beta_{j_0}$ for some $1 \le j_0 \le t$. Thus $\gamma_{i_0} \cdots c_3 \cdots c_{s-2}\delta_{j_0}$ is a reduced walk contained in Θ. This however implies that $\gamma_{i_0}, \delta_{j_0}$ point to the same direction in Θ, which is a contradiction. The proof is completed.

Acknowledgements. The authors gratefully acknowledge financial support from NSERC and FCAR. They also would like to express their gratitude to I. Assem and D. Happel for some useful discussions.

References

[1] I. Assem, *Tilted algebras of type* $\mathbf{A}_n$, Comm. Algebra **10** (1982), 2121-2139.

[2] I. Assem and A. Skowroński, *Iterated tilted algebras of type* $\tilde{\mathbf{A}}_n$, Math. Z. **195** (1987), 269-290.

[3] M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra, **15** (1987), 145-179.

[4] F. Coehlo and A. Skowroński, *On Auslander-Reiten components for quasi-tilted algebras*, Fund. Math. **149** (1996), 67-82.

[5] K. R. Fuller, *Biserial Rings*, Lecture Notes in Mathematics, **734** (Springer, Berlin, 1979), 64-90.

[6] P. Gabriel, *Auslander-Reiten sequences and representation-finite algebras*, Lecture Notes in Mathematics, **831** (Springer, Berlin, 1980), 1-71.

[7] E. L. Green, D. Happel and D. Zacharia, *Projective resolutions over algebras with zero relations*, Illinois J. Math. **29** (1985), 180-190.

[8] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Notes Series **119** (1988).

[9] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Memoirs Amer. Math. Soc. **575** (1996).

[10] D. Happel and C. M. Ringel, *Tilted Algebras*, Trans. Amer. Math. Soc. **274** (1982), 399-443.

[11] D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manucripta Math. **42** (1983), 221-243.

[12] F. Huard, *Tilted gentle algebras*, Comm. Algebra (to appear).

[13] J. A. de la Peña, *Representation-finite algebras whose Auslander-Reiten quiver is planar*, J. London Math. Soc. (2) **32** (1985), 62-74.

[14] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics **1099** (Springer, Berlin, 1984).

[15] O. Roldán, *Tilted algebras of types* $\tilde{\mathbf{A}}_n, \tilde{\mathbf{B}}_n, \tilde{\mathbf{C}}_n$ *and* $\widetilde{\mathbf{BC}}_n$, Ph.D. thesis, Carleton University (1983).

Département de Mathématiques et d'informatique
Université de Sherbrooke
Sherbrooke, Québec
Canada J1K 2R1