ALMOST SPLIT SEQUENCES AND APPROXIMATIONS

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Dedicated to the memory of Dieter Happel

Abstract. Let $\mathcal{A}$ be an exact category, that is, an extension-closed full subcategory of an abelian category. First, we give new characterizations of an almost split sequence in $\mathcal{A}$, which yields some necessary and sufficient conditions for $\mathcal{A}$ to have an almost split sequence with prescribed end terms. Then, we study when an almost split sequence in $\mathcal{A}$ induces an almost split sequence in an exact subcategory $\mathcal{C}$ of $\mathcal{A}$. In case $\mathcal{A}$ has almost split sequences and $\mathcal{C}$ is Ext-finite and Krull-Schmidt, we obtain a necessary and sufficient condition for $\mathcal{C}$ to have almost split sequences. Finally, we show some applications of these results.

Introduction

The Auslander-Reiten theory of almost split sequences has been playing a fundamental role in the representation theory of artin algebras with a great impact in other areas such as algebraic geometry and algebraic topology; see [6, 3, 15]. It is a long standing problem to determine which categories have almost split sequences. In the module category over an artin algebra, the existence of almost split sequences is derived from the Auslander-Reiten duality. In a general Hom-finite Krull-Schmidt exact category, Gabriel and Roiter showed that the existence of the Auslander-Reiten duality is necessary for the existence of almost split sequences; see [11], which is later proved to be sufficient by Lenzing and Zuazua in case the category is in addition Ext-finite; see [21]. On the other hand, it is natural to study when a subcategory of a category having almost split sequences has almost split sequences. A pioneering work in this direction by Auslander and Smalø shows that functorially finite subcategories of a module category have almost split sequences; see [8]. This has been further studied by other authors; see, for example, [17, 18]. Rather recently, Jørgensen considered the analogous problem for Hom-finite Krull-Schmidt triangulated categories, and proved that if the ambient category has almost split triangles, then the almost split triangles in a subcategory are linked to those in the ambient category by minimal approximations; see [16]. By passing through the homotopy category, this result is applied to obtain a necessary and sufficient condition for a subcategory of the module category of an artin algebra to have almost split sequences; see [22].

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Contrary to the cases mentioned above, almost split sequences appear naturally in abelian categories which are not Hom-finite; see, for example, [4, (4.1)] and [9, (2.8)]. This motivates us to study the two problems mentioned above in the most general setup. First, working with an arbitrary exact category, we shall characterize an almost split sequence in terms of linear forms on the stable endomorphism algebras of its end terms; see (2.2). This yields necessary and sufficient conditions for an exact category to have an almost split sequence with two prescribed end terms; see (2.3). Specializing to Hom-finite exact categories, we recover the result mentioned above by Gabriel-Roiter and Lenzing-Zuazua; see (2.6). Then, we investigate the relation between the almost split sequences in an exact category and those in its exact subcategories. The result says in particular that if the ambient category has almost split sequences, then the almost split sequences in a Hom-finite Krull-Schmidt exact subcategory are precisely the minimal projectively or injectively stable approximations of the almost split sequences in the ambient category; see (3.5) and (3.7). This is a strengthened analogous version, by means of a very different approach, of Jørgensen's result stated in [15]. Since our categories do not necessarily have projective or injective objects, one can not simply apply Jørgensen's result in our situation as is done in [22]. As an application, given any torsion theory in an exact category having almost split sequences, we show that the torsion subcategory has right almost split sequences and the torsion-free subcategory has left almost split sequences. Finally, we shall apply our results to give a new proof of the existence theorem of almost split sequences in the category of finitely presented representations of an infinite quiver.

1. Preliminaries

Throughout the paper, R stands for a commutative ring, which is not necessarily artinian unless otherwise explicitly stated. An R-category is a category in which the morphism sets are R-modules and the composition of morphisms is R-bilinear. Let $\mathcal{A}$ be an additive R-category, which will be called *Hom-finite* if its morphism modules are all of finite R-length. An idempotent endomorphism $e : X \to X$ is said to *split* in $\mathcal{A}$ if there exist morphisms $p : X \to Y$ and $q : Y \to X$ such that $e = qp$ and $p\,q = \mathbf{1}_Y$. Moreover, a non-zero object X is called *strongly indecomposable* if $\operatorname{End}_{\mathcal{A}}(X)$ is local, and *Krull-Schmidt* if it is a finite direct sum of strongly indecomposable objects. Now, $\mathcal{A}$ is called *Krull-Schmidt* if every non-zero objects is Krull-Schmidt. The following characterization of Krull-Schmidt objects is due to Gabriel and Roiter; see [11, (3.3)]. Since their approach is functorial, we present here an alternative elementary proof.

1.1. Proposition. *If X is an object in $\mathcal{A}$, then $X = X_1 \oplus \cdots \oplus X_n$ with X_i strongly indecomposable if and only if $\operatorname{End}_{\mathcal{A}}(X)$ is semiperfect and all its idempotents split; and in this case, each direct summand of X admits a decomposition as a direct sum of objects of a subfamily of $\{X_1, \ldots, X_n\}$, which is its unique (up to isomorphism and permutation) decomposition into a direct sum of indecomposable objects.*

Proof. Put $E = \operatorname{End}_{\mathcal{A}}(X)$. Suppose that E is semiperfect. Then E has a complete set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents such that the e_iEe_i are local; see [1, (27.6)]. If the idempotents in E split in $\mathcal{A}$, then $X = X_1 \oplus \cdots \oplus X_n$ with $\operatorname{End}_{\mathcal{A}}(X_i) \cong e_iEe_i$, that is, the X_i are strongly indecomposable.

Suppose now that $X = X_1 \oplus \cdots \oplus X_n$, with canonical injections $q_i : X_i \to X$, canonical projections $p_i : X \to X_i$, and local rings $\mathrm{End}_{\mathcal{A}}(X_i)$. Setting $e_i = q_i p_i$, we get a complete set $\{e_1, \ldots, e_n\}$ of orthogonal primitive idempotents in E such that $e_i E e_i \cong \mathrm{End}_{\mathcal{A}}(X_i)$. In particular, E is semiperfect; see [1, (27.6)]. Then, a classical result says that the complete sets of orthogonal primitive idempotents in E are pairwise conjugate up to permutation; see [14, (III.10.2)].

Let f be a non-zero idempotent in E. Since $E/\mathrm{rad}E$ is semi-simple, every non-zero idempotent in E is a sum of orthogonal primitive idempotents. In particular, there exists a complete set $\{f_1, \ldots, f_n\}$ of orthogonal primitive idempotents in E such that $f = f_1 + \cdots + f_r$, with $0 < r \le n$. Then, there exists a permutation σ and an invertible $a \in E$ such that $f_i = a e_{\sigma(i)} a^{-1}$, for $i = 1, \ldots, n$; Set $L = X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(r)}$, and

$$p = (p_{\sigma(1)}, \cdots, p_{\sigma(r)})^T a^{-1} : X \to L, \text{ and } q = a(q_{\sigma(1)}, \ldots, q_{\sigma(r)}) : L \to X.$$

Then $f = qp$ and $pq = \mathbf{1}_L$. That is, f splits.

Next, assume that M is a non-zero direct summand of X with a canonical injection $u : M \to X$ and a canonical projection $v : X \to M$. Set $f = uv$, a non-zero idempotent in E. As seen above, there exists a permutation σ and morphisms $p : X \to X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(r)}$ and $q : X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(r)} \to X$ such that $f = qp$ and $pq = \mathbf{1}_{X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(r)}}$. This yields an isomorphism $pu : M \to X_{\sigma(1)} \oplus \cdots \oplus X_{\sigma(r)}$.

Finally, in order to show the uniqueness of the decomposition of M, we need only to consider the case where $M = X$. Suppose that $X = Y_1 \oplus \cdots \oplus Y_m$, with canonical injections $u_i : Y_i \to X$, canonical projections $v_i : X \to Y_i$, and indecomposable objects Y_i. This yields a complete set $\{w_1, \ldots, w_m\}$ of orthogonal idempotents in E, where $w_i = u_i v_i$. Since the idempotents split in $\mathcal{A}$, the w_i are primitive. Therefore, $n = m$ and we may assume that there exists $b \in A$ such that $w_i = b e_i b^{-1}$, for $i = 1, \ldots, n$. Hence, $v_i b q_i : X_i \to Y_i$ is an isomorphism, $i = 1, \ldots, n$. The proof of the proposition is completed.

The following easy observation is needed for our later investigation.

1.2. Lemma. *Let $\mathcal{I}$ be an ideal of $\mathcal{A}$, and let X be an object in $\mathcal{A}$.*

(1) *If $\mathrm{End}_{\mathcal{A}}(X)$ is local, then $\mathcal{I}(X, X) = \mathrm{End}_{\mathcal{A}}(X)$ or $\mathcal{I}(X, X) \subseteq \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$.*

(2) *If $\mathcal{A}$ is Krull-Schmidt, then so is $\mathcal{A}/\mathcal{I}$.*

Proof. If $\mathrm{End}_{\mathcal{A}}(X)$ is local, then $\mathcal{I}(X, X) = \mathrm{End}_{\mathcal{A}}(X)$ or $\mathcal{I}(X, X) \subseteq \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$, and consequently, X is zero or strongly indecomposable in $\mathcal{A}/\mathcal{I}$. Now, if X is a finite direct sum of strongly indecomposable objects in $\mathcal{A}$, then it is zero or a finite direct sum of strongly indecomposable objects in $A/\mathcal{I}$. The proof of the lemma is completed.

We conclude this section with a brief recall of some classical terminology which will be needed for our later investigation. A morphism $f : X \to Y$ in $\mathcal{A}$ is *right minimal* if any endomorphism $g : X \to X$ such that $fg = f$ is an automorphism; and *left minimal* if any endomorphism $h : Y \to Y$ such that $hf = f$ is an automorphism. Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$, and let X be an object in $\mathcal{A}$. A morphism $f : M \to X$ with $M \in \mathcal{C}$ is called a *right $\mathcal{C}$-approximation* of X if the map $\mathrm{Hom}_{\mathcal{A}}(L, f) : \mathrm{Hom}_{\mathcal{A}}(L, M) \to \mathrm{Hom}_{\mathcal{A}}(L, X)$ is surjective for any $L \in \mathcal{C}$; and a *minimal right $\mathcal{C}$-approximation* if, in addition, f is right minimal. Dually, a morphism $g : X \to N$ with $N \in \mathcal{C}$ is called a *left $\mathcal{C}$-approximation* of X if the map

$\mathrm{Hom}_{\mathcal{A}}(f, L) : \mathrm{Hom}_{\mathcal{A}}(N, L) \to \mathrm{Hom}_{\mathcal{A}}(X, L)$ is surjective for any $L \in \mathcal{C}$; and a *minimal left $\mathcal{C}$-approximation* if, in addition, f is left minimal. Moreover, one says that $\mathcal{C}$ is *contravariantly finite* in $\mathcal{A}$ if every object in $\mathcal{A}$ has a right $\mathcal{C}$-approximation, *covariantly finite* in $\mathcal{A}$ if every object in $\mathcal{A}$ has a left $\mathcal{C}$-approximation, and *functorially finite* if it is both contravariantly finite and covariantly finite in $\mathcal{A}$; see [7].

2. Existence of almost split sequences

For the rest of the paper, let $\mathcal{A}$ stand for an *exact* R-category, that is, an extension-closed full subcategory of an abelian R-category $\mathscr{A}$. Clearly, the idempotents in $\mathcal{A}$ split if and only if $\mathcal{A}$ is closed under direct summands in $\mathscr{A}$. The objective of this section is to study the existence of almost split sequences in $\mathcal{A}$.

Let $\delta :\ 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a short exact sequence in $\mathcal{A}$. We shall call f a *proper monomorphism* and g a *proper epimorphism* in $\mathcal{A}$. Given any morphisms $u : X \to M$ and $v : N \to Z$ in $\mathcal{A}$, since $\mathcal{A}$ is extension-closed in $\mathscr{A}$, we have a pushout diagram

$$\begin{array}{lccccccccc} \delta : & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & & {\scriptstyle u}\downarrow & & \downarrow & & \| & & \\ u\delta : & 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

as well as a pullback diagram

$$\begin{array}{lccccccccc} \delta v : & 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle v} & & \\ \delta : & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

Thus, the equivalence classes of short exact sequences $0 \longrightarrow X \longrightarrow E \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$ form an abelian group $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ under Baer sum, which is an $\mathrm{End}_{\mathcal{A}}(X)$-$\mathrm{End}_{\mathcal{A}}(Z)$-bimodule under the multiplications illustrated in the above diagrams. Since $\mathrm{End}_{\mathcal{A}}(X)$ is an R-algebra, $\mathrm{Ext}^1_{\mathcal{A}}(X, Z)$ is an R-module. We shall say that $\mathcal{A}$ is *Ext-finite* if $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ is of finite R-length for all $X, Y \in \mathcal{A}$.

The stable categories of $\mathcal{A}$ introduced by Gabriel and Roiter are essential for our investigation; see [11, (9.2)], and also [21, (2.1)]. A morphism $u : X \to Y$ in $\mathcal{A}$ is called *injectively trivial* if the map

$$\mathrm{Ext}^1_{\mathcal{A}}(L, u) :\ \mathrm{Ext}^1_{\mathcal{A}}(L, X) \to \mathrm{Ext}^1_{\mathcal{A}}(L, Y) : \eta \mapsto u\eta$$

is zero for any $L \in \mathcal{A}$, or equivalently, u factors through any proper monomorphism $v : X \to M$ in $\mathcal{A}$. Dually, $u : X \to Y$ is called *projectively trivial* if the map

$$\mathrm{Ext}^1_{\mathcal{A}}(u, L) :\ \mathrm{Ext}^1_{\mathcal{A}}(Y, L) \to \mathrm{Ext}^1_{\mathcal{A}}(X, L) : \zeta \mapsto \zeta u$$

is zero for any $L \in \mathcal{A}$, or equivalently, u factors through any proper epimorphism $w : N \to Y$ in $\mathcal{A}$. It is easy to verify that the injectively trivial morphisms in $\mathcal{A}$ form an ideal, denoted as $I_{\mathcal{A}}$; and the projectively trivial morphisms form an ideal, as denoted as $P_{\mathcal{A}}$. Now, the quotient category $\overline{\mathcal{A}} = \mathcal{A}/I_{\mathcal{A}}$ is called the *injectively stable category* of $\mathcal{A}$, while $\underline{\mathcal{A}} = \mathcal{A}/P_{\mathcal{A}}$ is called the *projectively stable category.* In the sequel, for a morphism $u : X \to Y$ in $\mathcal{A}$, we shall denote by $\overline{u}$ and $\underline{u}$ its images in $\mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y)$ and $\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y)$, respectively. Finally, an object $X \in \mathcal{A}$ is called *Ext-injective* if every proper monomorphism $f : X \to Y$ is a section; and

Ext-projective if every proper epimorphism $g : Y \to X$ is a retraction. It is easy to see that X is Ext-injective if and only if $\mathbb{1}_X$ is injectively trivial, or equivalently, X is zero in $\overline{\mathcal{A}}$. Dually, X is Ext-projective if and only if $\mathbb{1}_X$ is projectively trivial, or equivalently, X is zero in $\underline{\mathcal{A}}$.

Next, we recall from [5] some terminology and facts for the Auslander-Reiten theory. Let $f : X \to Y$ be a morphism. One says that f is *right almost split* if f is not a retraction and every non-retraction morphism $g : M \to Y$ factors through f; and *minimal right almost split* if f is right minimal and right almost split. In a dual manner, one defines f to be *left almost split* and *minimal left almost split.* Note that if $f : X \to Y$ is left or right almost split, then X or Y is strongly indecomposable, respectively; see [5]. A short exact sequence

$$\delta :\ 0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0$$

is called *almost split* if f is minimal left almost split and g is minimal right almost split; see [5]. In this case, both the $\mathrm{End}_{\mathcal{A}}(X)$-socle and the $\mathrm{End}_{\mathcal{A}}(Z)$-socle of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ are simple generated by δ. Moreover, since δ is unique up to isomorphism for X and for Z, we may write $X = \tau_{_\mathcal{A}} Z$ and $Z = \tau^-_{_\mathcal{A}} X$. We shall say that $\mathcal{A}$ has *right almost split sequences* if every indecomposable object is either Ext-projective or the ending term of an almost split sequence, $\mathcal{A}$ has *left almost split sequences* if every indecomposable object is either Ext-injective or the starting term of an almost split sequence, and $\mathcal{A}$ has *almost split sequences* if it has both left and right almost split sequences.

From now on, fix an injective co-generator I for the category $\mathrm{Mod}R$ of all R-modules, which will be minimal if R is artinian. Then we have an exact endofunctor $D = \mathrm{Hom}_R(-,\ I) : \mathrm{Mod}R \to \mathrm{Mod}R$. For $U, V \in \mathrm{Mod}R$, an R-bilinear form $<\,,\,>: U \times V \to I$ is called *non-degenerate* provided that, for any non-zero element $u \in U$, there exists some $v \in V$ such that $<u, v> \neq 0$, and for any non-zero element $v \in V$, there exists some $u \in U$ such that $<u, v> \neq 0$. Observe that every R-linear form $\varphi : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to I$ determines, for each $L \in \mathcal{A}$, two R-bilinear forms:

$$<\,,\,>_\varphi :\ \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I : (\bar{f}, \eta) \mapsto \varphi(f\eta),$$

and

$$_\varphi<\,,\,>:\ \mathrm{Ext}^1_{\mathcal{A}}(L, X) \times \mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L) \to I : (\zeta,\, \underline{g}) \mapsto \varphi(\zeta g).$$

On the other hand, if δ is a non-zero extension in $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$, then there exists always an R-linear form $\varphi : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to I$ such that $\varphi(\delta) \neq 0$. We are now ready to state the following result of Gabriel and Roiter, which is implicitly stated in [11, (9.3)]; see also [21, (3.1)]. We include their proof for the reader's convenience,

2.1. Proposition. *Let* $\delta :\ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be an almost split sequence in* $\mathcal{A}$*, and let* $\varphi : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to I$ *be an* R*-linear form. If* $\varphi(\delta) \neq 0$*, then the* R*-bilinear forms*

$$<\,,\,>_\varphi :\ \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I$$

and

$$_\varphi<\,,\,>:\ \mathrm{Ext}^1_{\mathcal{A}}(L, X) \times \mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L) \to I$$

are both non-degenerate, for every $L \in \mathcal{A}$.

Proof. Suppose that $\varphi(\delta) \neq 0$ and $L \in \mathcal{A}$. We shall prove only that ${}_{\varphi}<\ ,\ >$ is non-degenerate. Let $g : Z \to L$ be a morphism in $\mathcal{A}$ which is not projectively trivial. Then $\mathcal{A}$ admits a pullback diagram

$$\begin{array}{lccccccccc} \eta: & 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\ \zeta: & 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

with non-split rows. Since δ is almost split, there exists a pushout diagram

$$\begin{array}{lccccccccc} \eta: & 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle h} & & \downarrow & & \| & & \\ \delta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$. This yields $\delta = h\eta = h(\zeta g) = (h\zeta)g$. As a consequence, $h\zeta \in \mathrm{Ext}^1_{\mathcal{A}}(L, X)$ is such that ${}_{\varphi}< h\zeta, g >= \varphi((h\zeta)g) = \varphi(\delta) \neq 0$. On the other hand, consider a non-split short exact sequence $\zeta :\ 0 \longrightarrow X \longrightarrow E \longrightarrow L \longrightarrow 0$ in $\mathcal{A}$. Since δ is almost split, there exists a pullback diagram

$$\begin{array}{lccccccccc} \delta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\ \zeta: & 0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$. Thus $\underline{g} \in \mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L)$ is such that ${}_{\varphi}<\zeta,\ \underline{g}>= \varphi(\zeta g) = \varphi(\delta) \neq 0$. The proof of the proposition is completed.

Our first result will be a characterization of an almost split sequence. We need to introduce some terminology. Let $F : \mathcal{A} \to \mathrm{Mod}R$ and $G : \mathcal{A} \to \mathrm{Mod}R$ be covariant or contravariant R-linear functors. A *functorial monomorphism* $\phi : F \to G$ is a natural transformation with $\phi_X : F(X) \to G(X)$ being injective for all $X \in \mathcal{A}$. Moreover, if Λ is an R-algebra, then a non-zero R-linear form $\phi : \Lambda \to I$ is called *almost vanishing* if it vanishes on the Jacobson radical of Λ.

2.2. Theorem. *Let $\mathcal{A}$ be an exact R-category, which has a short exact sequence $\delta :\ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ with X, Z being strongly indecomposable. The following statements are equivalent.*

(1) *The sequence δ is an almost split sequence in $\mathcal{A}$.*
(2) *There exists a functorial monomorphism $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$ such that $\phi_X(\delta)$ is almost vanishing on $\mathrm{End}_{\overline{\mathcal{A}}}(X)$.*
(3) *There exists a functorial monomorphism $\psi : \mathrm{Ext}^1_{\mathcal{A}}(-, X) \to D\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, -)$ such that $\psi_Z(\delta)$ is almost vanishing on $\mathrm{End}_{\underline{\mathcal{A}}}(Z)$.*

Proof. We shall prove only the equivalence of Statements (1) and (2). Since $\delta \neq 0$ in each of the statements, we may assume that X is not Ext-injective, that is, X is non-zero in $\overline{\mathcal{A}}$. By Lemma 1.2(1), $\mathrm{rad}(\mathrm{End}_{\overline{\mathcal{A}}}(X)) = \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))/I_{\mathcal{A}}(X, X)$.

Assume first that δ is an almost split sequence. In particular, there exists an R-linear form $\varphi : \mathrm{Ext}^1(Z, X) \to I$ such that $\varphi(\delta) \neq 0$. Fix $L \in \mathcal{A}$. By Proposition 2.1, we have a non-degenerate bilinear form

$$<\ ,\ >_{\varphi}:\ \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I : (\bar{f}, \eta) \mapsto \varphi(f\eta).$$

This induces an R-linear monomorphism

$$\phi_L : \mathrm{Ext}^1_{\mathcal{A}}(Z,\ L) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(L,\ X) : \eta \mapsto <-\,, \eta>_{\varphi},$$

which is clearly natural in L. Since ϕ_X is injective, $\phi_X(\delta) \neq 0$. If $\bar{f} \in \mathrm{rad}(\mathrm{End}_{\overline{\mathcal{A}}}(X))$, then $f \in \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$. Since δ is almost split, we have $f\delta = 0$. As a consequence, $\phi_X(\delta)(f) = <f, \delta>_{\varphi} = \varphi(f\delta) = 0$. Thus, $\phi_X(\delta)$ is almost vanishing on $\mathrm{End}_{\overline{\mathcal{A}}}(X)$.

Conversely, let $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$ be a functorial monomorphism such that $\phi_X(\delta)$ is almost vanishing on $\mathrm{End}_{\overline{\mathcal{A}}}(X)$. Then, $\delta \neq 0$. Let $u : X \to L$ be a morphism in $\mathcal{A}$ which is not a section. For any morphism $v : L \to X$, we have $vu \in \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$, and hence, $\bar{v}\bar{u} \in \mathrm{rad}(\mathrm{End}_{\overline{\mathcal{A}}}(X))$. Thus $\phi_X(\delta)(\bar{v}\bar{u}) = 0$, that is, $(D\mathrm{Hom}_{\overline{\mathcal{A}}}(u, X) \circ \phi_X)(\delta) = 0$. In view of the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{A}}(Z, X) & \xrightarrow{\mathrm{Ext}^1_{\mathcal{A}}(Z,u)} & \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \\ {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \phi_L} \\ D\mathrm{Hom}_{\overline{\mathcal{A}}}(X, X) & \xrightarrow{D\mathrm{Hom}_{\overline{\mathcal{A}}}(u,X)} & D\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X), \end{array}$$

we see that $(\phi_L \circ \mathrm{Ext}^1_{\mathcal{A}}(Z, u))(\delta) = 0$. Since ϕ_L is injective, $u\delta = \mathrm{Ext}^1_{\mathcal{A}}(Z, u)(\delta) = 0$. That is, u factors through the monomorphism $X \to Y$ in δ. As a consequence, δ is an almost split sequence; see [2, (4.4)]. The proof of the theorem is completed.

If $X, Y \in \mathcal{A}$, then $D\mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y)$ is an $\mathrm{End}_{\mathcal{A}}(X)$-$\mathrm{End}_{\mathcal{A}}(Y)$-bimodule with multiplications defined, for $f \in \mathrm{End}_{\mathcal{A}}(X)$, $\theta \in D\mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y)$, $g \in \mathrm{End}_{\mathcal{A}}(Y)$, by

$$f\theta g : \mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y) \to I : \bar{h} \mapsto \theta(\overline{ghf}).$$

Similarly, $D\mathrm{Hom}_{\underline{\mathcal{A}}}(X, Y)$ is an $\mathrm{End}_{\mathcal{A}}(X)$-$\mathrm{End}_{\mathcal{A}}(Y)$-bimodule. As promised, we have the following existence theorem of an almost split sequence with prescribed end terms.

2.3. THEOREM. *Let $\mathcal{A}$ be an exact R-category, and let X, Z be strongly indecomposable objects in $\mathcal{A}$. The following statements are equivalent.*

(1) *There exists an almost split sequence* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *in $\mathcal{A}$.*

(2) *The* $\mathrm{End}_{\mathcal{A}}(X)$*-socle of* $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ *is non-zero and there exists a functorial monomorphism* $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$.

(3) *The* $\mathrm{End}_{\mathcal{A}}(Z)$*-socle of* $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ *is non-zero and there exists a functorial monomorphism* $\psi : \mathrm{Ext}^1_{\mathcal{A}}(-, X) \to D\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, -)$.

Proof. We shall prove only the equivalences of Statements (1) and (2). Assume first that $\mathcal{A}$ has an almost split sequence $\delta :\ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$. By Theorem 2.2, there exists a functorial monomorphism $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$. Being almost split, δ is a non-zero element in the $\mathrm{End}_{\mathcal{A}}(X)$-socle of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$.

Conversely, suppose that $\delta :\ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a non-zero extension lying in the $\mathrm{End}_{\mathcal{A}}(X)$-socle of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$. In particular, X is not Ext-injective and Z is not Ext-projective. Let $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$ be a functorial monomorphism. Since ϕ is natural, $\phi_X : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to D\mathrm{End}_{\overline{\mathcal{A}}}(X)$ is $\mathrm{End}_{\mathcal{A}}(X)$-linear. Hence, $\theta = \phi_X(\delta)$ is a non-zero element of $D\mathrm{End}_{\overline{\mathcal{A}}}(X)$, which is annihilated by $\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$. If $\bar{f} \in \mathrm{rad}(\mathrm{End}_{\overline{\mathcal{A}}}(X))$, then $f \in \mathrm{rad}(\mathrm{End}_{\mathcal{A}}(X))$ by Lemma 1.2(1), and hence, $\theta(\bar{f}) = (f\theta)(\bar{\mathbf{1}}) = 0$. That is, θ is almost vanishing. By Theorem 2.2, δ is almost split in $\mathcal{A}$. The proof of the theorem is completed.

REMARK. Let Λ be an artin R-algebra with R being artinian. If $M \in \mathrm{mod}\Lambda$ is indecomposable and non-projective, then $\mathrm{DTr}\, M \in \mathrm{mod}\Lambda$ is indecomposable and non-injective with a natural isomorphism $D\mathrm{Ext}^1_\Lambda(M, L) \cong \overline{\mathrm{Hom}}_\Lambda(L, \mathrm{DTr}\, M)$ for every module $L \in \mathrm{Mod}\Lambda$; see [2, (III.4.3)], and also [19]. As a consequence, we have a natural monomorphism $\phi_L : \mathrm{Ext}^1_\Lambda(M, L) \to D\overline{\mathrm{Hom}}_\Lambda(L, \mathrm{DTr}\, M)$, for every module $L \in \mathrm{Mod}\Lambda$. By Theorem 2.3(1), there exists an almost split sequence

$$0 \longrightarrow \mathrm{DTr}\, M \longrightarrow N \longrightarrow M \longrightarrow 0$$

in $\mathrm{Mod}\Lambda$, which is also an almost split sequence in $\mathrm{mod}\Lambda$. This is a well known result of Auslander and Reiten; see [4, (4.1)]. Note that ϕ_L is not an isomorphism in general. This shows that Theorem 2.3 is a genuine generalization of the Auslander-Reiten duality.

For the rest of this section, we shall concentrate on the case where R is artinian and I is the minimal injective co-generator for $\mathrm{Mod}R$. In this case, there exists a well known duality

$$D = \mathrm{Hom}_R(-, I) : \mathrm{mod}R \to \mathrm{mod}R,$$

where $\mathrm{mod}R$ denotes the category of finitely generated R-modules. The following easy observation is useful in our study.

2.4. LEMMA. *Let R be artinian, and let $<\ ,\ >: U \times V \to I$ be a non-degenerate R-bilinear form, where $U, V \in \mathrm{Mod}R$. If U or V is of finite R-length, then we have two R-linear isomorphisms*

$$V \to DU : v \mapsto <-\,, v> \quad \textit{and} \quad U \to DV : u \mapsto <u, -> .$$

Proof. Since $<\ ,\ >$ is non-degenerate, we have two R-linear monomorphisms:

$$\phi : V \to DU : v \mapsto <-\,, v> \quad \text{and} \quad \psi : U \to DV : u \mapsto <u, -> .$$

Suppose that V has finite R-length $\ell_R(V)$. Then $\ell_R(DV) = \ell_R(V)$. Since ψ is injective, $\ell_R(U) \le \ell_R(V)$. On the other hand, since ϕ is injective, we have $\ell_R(V) \le \ell_R(DU) = \ell_R(U) \le \ell_R(V)$. This yields $\ell_R(V) = \ell_R(DU)$, and hence, $\ell_R(U) = \ell_R(DV)$. As a consequence, ϕ and ψ are isomorphisms. Similarly, the result holds true if $\ell_R(U)$ is finite. The proof of the lemma is completed.

2.5. LEMMA. *Let $\mathcal{A}$ be an exact R-category where R is artinian, which has an almost split sequence* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$.

(1) *If $L \in \mathcal{A}$, then $\mathrm{Ext}^1_{\mathcal{A}}(Z, L)$ is of finite R-length if and only if so is $\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$; and in this case, $\mathrm{Ext}^1_{\mathcal{A}}(Z, L) \cong D\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$.*
(2) *If $L \in \mathcal{A}$, then $\mathrm{Ext}^1_{\mathcal{A}}(L, X)$ is of finite R-length if and only if so is $\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L)$; and in this case, $\mathrm{Ext}^1_{\mathcal{A}}(L, X) \cong D\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L)$.*

Proof. We shall prove only Statement (1). For any $L \in \mathcal{A}$, by Proposition 2.1, there exists a non-degenerate R-bilinear form

$$<\ ,\ >: \ \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I.$$

If $\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$ or $\mathrm{Ext}^1_{\mathcal{A}}(Z, L)$ is of finite R-length, then it follows from Lemma 2.4 that $\mathrm{Ext}^1_{\mathcal{A}}(Z, L) \cong D\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$. The proof of the lemma is completed.

The following result is a local version, but under weaker hypotheses, of the main result stated in [21].

2.6. Theorem. *Let $\mathcal{A}$ be an exact R-category where R is artinian, and let $X, Z \in \mathcal{A}$ be strongly indecomposable with X not Ext-injective and Z not Ext-projective.*

(1) *If* $\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \in \mathrm{mod}R$ *for all* $L \in \mathcal{A}$, *then* $\mathcal{A}$ *has an almost split sequence* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *if and only if* $\mathrm{Ext}^1_{\mathcal{A}}(Z, -) \cong D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$.

(2) *If* $\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, L) \in \mathrm{mod}R$ *for all* $L \in \mathcal{A}$, *then* $\mathcal{A}$ *has an almost split sequence* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *if and only if* $\mathrm{Ext}^1_{\mathcal{A}}(-, X) \cong D\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, -)$.

Proof. We shall prove only Statement (1). Suppose that $\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$ is of finite R-length for any $L \in \mathcal{A}$. Let $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$ be a functorial isomorphism. Since Z is not Ext-projective, $\mathrm{End}_{\overline{\mathcal{A}}}(X) \neq 0$. Since ϕ_X is bijective, $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is non-zero of finite R-length. In particular, the $\mathrm{End}^1_{\mathcal{A}}(X)$-socle of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is non-zero. By Theorem 2.3, $\mathcal{A}$ has a desired almost split sequence.

Conversely, let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an almost split sequence in $\mathcal{A}$. By Theorem 2.2, we have a functorial monomorphism $\phi : \mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)$. For each $L \in \mathcal{A}$, by Lemma 2.5(1), $\mathrm{Ext}^1_{\mathcal{A}}(Z, L) \cong D\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$, and hence, the monomorphism $\phi_L : \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(L, X)$ is an isomorphism. The proof of the theorem is completed.

Remark. As an example, the situations described in Theorem 2.6 occur in the category of locally finite dimensional representations of a strongly locally finite quiver; see (4.1).

We conclude this section with the following easy consequence.

2.7. Corollary. *Let $\mathcal{A}$ be a Krull-Schmidt exact R-category where R is artinian. If $\mathcal{A}$ has almost split sequences, then $\mathcal{A}$ is Ext-finite if and only if $\overline{\mathcal{A}}$ is Hom-finite, if and only if $\underline{\mathcal{A}}$ is Hom-finite.*

Proof. Let $X, Y \in \mathcal{A}$ be indecomposable. Suppose first that $\bar{\mathcal{A}}$ is Hom-finite. If X is Ext-projective, then $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = 0$. Otherwise, since $\mathrm{Hom}_{\overline{\mathcal{A}}}(Y, \tau_{\mathcal{A}} X)$ is of finite R-length, so is $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ by Lemma 2.5(1). Thus $\mathcal{A}$ is Ext-finite.

Suppose next that $\mathcal{A}$ is Ext-finite. If Y is Ext-injective, $\mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y) = 0$. Otherwise, since $\mathrm{Ext}^1_{\mathcal{A}}(\tau^-_{\mathcal{A}} Y, X)$ is of finite R-length, so is $\mathrm{Hom}_{\overline{\mathcal{A}}}(X, Y)$ by Lemma 2.5(1). This shows that $\overline{\mathcal{A}}$ is Hom-finite. Similarly, $\mathcal{A}$ is Ext-finite if and only if $\underline{\mathcal{A}}$ is Hom-finite. The proof of the corollary is completed.

3. Minimal approximations

Throughout this section, $\mathcal{A}$ stands for an exact R-category, and $\mathcal{C}$ for an exact subcategory of $\mathcal{A}$, that is, $\mathcal{C}$ is an extension-closed full subcategory of $\mathcal{A}$. The objective of this section is to study when an almost split sequence in $\mathcal{A}$ induces an almost split sequence in $\mathcal{C}$. For this purpose, we need to introduce some notation and terminology. Let $\widetilde{\mathcal{C}}$ and $\underset{\sim}{\mathcal{C}}$ stand for the full subcategories generated by the objects in $\mathcal{C}$ of $\bar{\mathcal{A}}$ and $\underline{\mathcal{A}}$, respectively. Observe that the injectively stable category $\overline{\mathcal{C}}$ of $\mathcal{C}$ is a quotient category of $\widetilde{\mathcal{C}}$, while the projectively stable category $\underline{\mathcal{C}}$ is a quotient category of $\underset{\sim}{\mathcal{C}}$.

3.1. Definition. Let $X \in \mathcal{A}$. A morphism $f : M \to X$ in $\mathcal{A}$ with $M \in \mathcal{C}$ is called a *right injectively stable $\mathcal{C}$-approximation* of X if $\bar{f}$ is a right $\widetilde{\mathcal{C}}$-approximation of

X in $\bar{\mathcal{A}}$; and a *minimal right injectively stable $\mathcal{C}$-approximation* if, in addition, $\bar{f}$ is right minimal in $\bar{\mathcal{A}}$ and M has no non-zero summand which is Ext-injective in $\mathcal{A}$.

In a dual manner we define, for an object in $\mathcal{A}$, the notions of *left projectively stable $\mathcal{C}$-approximation* and *minimal left projectively stable $\mathcal{C}$-approximation*.

3.2. LEMMA. *Let $X \in \mathcal{A}$ with a right injectively stable $\mathcal{C}$-approximation $f : M \to X$. If M is Krull-Schmidt, then f decomposes as $f = (g, h) : N \oplus L \to X$, where g is a minimal right injectively stable $\mathcal{C}$-approximation of X.*

Proof. Let M be Krull-Schmidt. Then $f = (f_1, \ldots, f_r) : M = M_1 \oplus \cdots \oplus M_r \to X$, where the M_i are strongly indecomposable in $\mathcal{C}$. If f is injectively trivial in $\mathcal{A}$, then $0 : 0 \to X$ is a minimal injectively stable $\mathcal{C}$-approximation of X. Otherwise, we may assume that there exists some $1 \le s \le r$ such that M_i is Ext-injective in $\mathcal{A}$ if and only if $s < i \le r$. Put $U = M_1 \oplus \cdots \oplus M_s$ and $u = (f_1, \ldots, f_s) : U \to X$. Then $\bar{u}$ is a right $\widetilde{\mathcal{C}}$-approximation of X in $\bar{\mathcal{A}}$. By Lemma 1.2, the M_i with $1 \le i \le s$ are strongly indecomposable in $\bar{\mathcal{A}}$. Hence, by Proposition 1.1, the idempotents in $\mathrm{End}_{\bar{\mathcal{A}}}(U)$ split in $\bar{\mathcal{A}}$. Therefore, there exists a decomposition $\bar{u} = (\bar{v}, \bar{0}) : U = V \oplus W \to X$, where $\bar{v}$ is right minimal in $\bar{\mathcal{A}}$; see [20, (1.4)]. Then $\bar{v} : V \to X$ is a minimal right $\widetilde{\mathcal{C}}$-approximation of X in $\bar{\mathcal{A}}$. Since V is a direct summand of M in $\bar{\mathcal{A}}$, by Proposition 1.1, we may assume that $\bar{\mathcal{A}}$ has an isomorphism $\bar{p} : N \to V$, where $N = M_1 \oplus \cdots \oplus M_t$ for some $1 \le t \le s$. Setting $g = vp$, we get a minimal right $\widetilde{\mathcal{C}}$-approximation $\bar{g} : N \to X$ of X in $\bar{\mathcal{A}}$. Therefore, $g : N \to X$ is a minimal right injectively stable $\mathcal{C}$-approximation of X. The proof of the lemma is completed.

The following result characterizes the Ext-projective objects and the Ext-injective objects in $\mathcal{C}$ which admit an almost split sequence in $\mathcal{A}$.

3.3. LEMMA. *Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an almost split sequence in $\mathcal{A}$.*

(1) *If $Z \in \mathcal{C}$, then Z is Ext-projective in $\mathcal{C}$ if and only if $0 \to X$ is a right injectively stable $\mathcal{C}$-approximation of X.*
(2) *If $X \in \mathcal{C}$, then X is Ext-injective in $\mathcal{C}$ if and only if $Z \to 0$ is a left projectively stable $\mathcal{C}$-approximation of Z.*

Proof. We shall prove only Statement (1). For each $L \in \mathcal{C}$, by Proposition 2.1, there exists a non-degenerate R-bilinear form

$$<\ ,\ >: \mathrm{Hom}_{\bar{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I.$$

If $Z \in \mathcal{C}$, then $\mathrm{Ext}^1_{\mathcal{A}}(Z, L) = \mathrm{Ext}^1_{\mathcal{C}}(Z, L)$. Therefore, Z is Ext-projective in $\mathcal{C}$ if and only if $\mathrm{Ext}^1_{\mathcal{C}}(Z, L) = 0$ for all $L \in \mathcal{C}$, if and only if $\mathrm{Hom}_{\bar{\mathcal{A}}}(L, X) = 0$ for all $L \in \mathcal{C}$, that is, $0 \to X$ is a right injectively stable $\mathcal{C}$-approximation of X. The proof of the lemma is completed.

We shall now show that minimal $\mathcal{C}$-approximations of almost split sequences in $\mathcal{A}$ are almost split sequences in $\mathcal{C}$. For this purpose, the following preparatory lemma will be useful.

3.4. LEMMA. *Let $Z \in \mathcal{C}$ with an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$, and let X have a minimal right injectively stable $\mathcal{C}$-approximation $f : M \to X$.*

(1) *For $L \in \mathcal{C}$, the map $\mathrm{Ext}^1_{\mathcal{A}}(L, f) : \mathrm{Ext}^1_{\mathcal{A}}(L, M) \to \mathrm{Ext}^1_{\mathcal{A}}(L, X)$ is injective.*
(2) *If $u : L \to M$ lies in $\mathcal{C}$, then $u \in I_{\mathcal{C}}(L, M)$ if and only if $fu \in I_{\mathcal{A}}(L, X)$.*

(3) *If Z is not Ext-projective in $\mathcal{C}$, then M is indecomposable.*

Proof. (1) Let $L \in \mathcal{C}$, and consider a pushout diagram

$$\begin{array}{lccccccccc} \eta: & 0 & \longrightarrow & M & \xrightarrow{r} & E & \longrightarrow & L & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\ \zeta: & 0 & \longrightarrow & X & \xrightarrow{s} & F & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$. If ζ splits, then $ts = \mathbb{1}$ for some $t : F \to X$, and hence $f = (tg)r$. Since $E \in \mathcal{C}$, there exists some $h : E \to M$ such that $\overline{tg} = \overline{fh}$. This gives rise to $\bar{f} = \bar{f} \cdot \overline{hr}$. Since $\bar{f}$ is right minimal in $\bar{\mathcal{A}}$, we see that $\overline{hr}$ is an automorphism of M in $\bar{\mathcal{A}}$. Therefore, $\overline{wr} = \bar{\mathbb{1}}$, for some $w : E \to M$. That is, $\mathbb{1} - wr$ is injectively trivial in $\mathcal{A}$. In particular, $\mathbb{1} - wr$ factors through r. Then r is a section, that is, η splits.

(2) Let $u : L \to M$ be a morphism in $\mathcal{C}$. Assume first that fu is not injectively trivial in $\mathcal{A}$. Let $\varphi : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to I$ be an R-linear form such that $\varphi(\delta) \neq 0$. In view of Proposition 2.1, there exists $\zeta \in \mathrm{Ext}^1_{\mathcal{A}}(Z, L)$ such that $\varphi(fu\zeta) \neq 0$. In particular, $u\,\zeta \neq 0$. Since ζ lies in $\mathcal{C}$, we see that $u \notin I_{\mathcal{C}}(L, M)$. Suppose conversely that fu is injectively trivial in $\mathcal{A}$. If $\eta : \; 0 \longrightarrow L \longrightarrow E \longrightarrow N \longrightarrow 0\;$ is a short exact sequence in $\mathcal{C}$, then $(fu)\eta = 0$, that is, $f(u\eta) = 0$. By Statement (1), we have $u\eta = 0$. Therefore, $u \in I_{\mathcal{C}}(L, M)$.

(3) Since the right $\mathrm{End}_{\mathcal{A}}(Z)$-module $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ has a simple socle, every non-zero $\mathrm{End}_{\mathcal{A}}(Z)$-submodule of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is indecomposable. Suppose that Z is not Ext-projective in $\mathcal{C}$. In particular, it is not Ext-projective in $\mathcal{A}$. By Lemma 3.3(1), f is not injectively trivial in $\mathcal{A}$. By Proposition 2.1, $\mathrm{Ext}^1_{\mathcal{A}}(Z, M) \neq 0$. Observe that $\mathrm{Ext}^1_{\mathcal{A}}(Z, f) : \mathrm{Ext}^1_{\mathcal{A}}(Z, M) \to \mathrm{Ext}^1_{\mathcal{A}}(Z, X)$ is $\mathrm{End}_{\mathcal{A}}(Z)$-linear and injective by Statement (1). Thus, $\mathrm{Ext}^1_{\mathcal{A}}(Z, M)$ is isomorphic to a non-zero $\mathrm{End}_{\mathcal{A}}(Z)$-submodule of $\mathrm{Ext}^1_{\mathcal{A}}(Z, X)$, and hence, it is an indecomposable right $\mathrm{End}_{\mathcal{A}}(Z)$-module. Assume that $M = M_1 \oplus M_2$, with non-zero injections $q_i : M_i \to M$, $i = 1, 2$. By the hypothesis, M_1 and M_2 are non-zero in $\bar{\mathcal{A}}$. Since $\bar{f}$ is right minimal in $\bar{\mathcal{A}}$, we have $\overline{fq_i} \neq \bar{0}$, and by Proposition 2.1, $\mathrm{Ext}^1_{\mathcal{A}}(Z, M_i) \neq 0$, $i = 1, 2$. Since

$$\mathrm{Ext}^1_{\mathcal{A}}(Z, M) \cong \mathrm{Ext}^1_{\mathcal{A}}(Z, M_1) \oplus \mathrm{Ext}^1_{\mathcal{A}}(Z, M_2),$$

we get a desired contradiction. The proof of the lemma is completed.

3.5. Proposition. *Let Z be an object in $\mathcal{C}$, which admits an almost split sequence $\delta : \; 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0\;$ in $\mathcal{A}$. Suppose that X has a minimal right injectively stable $\mathcal{C}$-approximation $f : M \to X$. If M is Krull-Schmidt in $\mathcal{C}$, then $\mathcal{A}$ has a pushout diagram*

$$\begin{array}{lccccccccc} \eta: & 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ \delta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0; \end{array}$$

and in any such pushout diagram, η is an almost split sequence in $\mathcal{C}$.

Proof. Assume that M is Krull-Schmidt in $\mathcal{C}$. Being non-zero, by definition, M is not Ext-injective in $\mathcal{A}$. That is, M is non-zero in $\bar{\mathcal{A}}$. Being right minimal, $\bar{f}$ is non-zero in $\bar{\mathcal{A}}$. By Lemma 3.3(1), Z is not Ext-projective in $\mathcal{C}$. In view of Lemma 3.4(3), M is strongly indecomposable. Now, for each $L \in \mathcal{C}$, we deduce from Lemma

3.4(2) that there exists an exact sequence

$$0 \longrightarrow I_{\mathcal{C}}(L, M) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(L, M) \xrightarrow{f^*} \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \longrightarrow 0$$

of R-modules, where f^* is the map induced from the left multiplication by f. This yields an R-linear isomorphism

$$\mathrm{Hom}_{\overline{\mathcal{C}}}(L, M) \to \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) : \bar{u} \mapsto \overline{fu},$$

which is clearly natural in L. That is, we have a functorial isomorphism

$$D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X)|_{\mathcal{C}} \to D\mathrm{Hom}_{\overline{\mathcal{C}}}(-, M).$$

On the other hand, by Theorem 2.2, there exists a functorial monomorphism

$$\mathrm{Ext}^1_{\mathcal{A}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{A}}}(-, X).$$

Since $\mathrm{Ext}^1_{\mathcal{C}}(Z, -) = \mathrm{Ext}^1_{\mathcal{A}}(Z, -)|_{\mathcal{C}}$, we get a functorial monomorphism

$$\phi : \mathrm{Ext}^1_{\mathcal{C}}(Z, -) \to D\mathrm{Hom}_{\overline{\mathcal{C}}}(-, M).$$

Observe that f is not injectively trivial. By Proposition 2.1, there exists a non-zero extension $\zeta \in \mathrm{Ext}^1_{\mathcal{A}}(Z, M)$ such that $f\zeta \neq 0$. Since δ is almost split in $\mathcal{A}$, there exists $g \in \mathrm{End}_{\mathcal{A}}(Z)$ such that $\delta = (f\zeta)g = f(\zeta g)$. This establishes the existence of a commutative diagram as stated in the proposition.

Now, let $\eta \in \mathrm{Ext}^1_{\mathcal{C}}(M, Z)$ be such that $f\eta = \delta$. Suppose that $u\eta \neq 0$ for some $u \in \mathrm{rad}(\mathrm{End}_{\mathcal{C}}(M))$. Since δ is almost split, there exists some $v : M \to X$ such that $v(u\eta) = \delta$. Since $\bar{f}$ is a right $\widetilde{\mathcal{C}}$-approximation of X, there exists some $w : M \to M$ such that $\overline{v} = \overline{fw}$. This yields $f\eta = \delta = f(wu\eta)$. By Lemma 3.4(1), $\eta = (wu)\eta$. Since $wu \in \mathrm{rad}(\mathrm{End}_{\mathcal{C}}(M))$, we get $\eta = 0$, a contradiction. This proves that η lies in the $\mathrm{End}_{\mathcal{C}}(M)$-socle of $\mathrm{Ext}^1_{\mathcal{A}}(M, Z)$. Since ϕ_M is an $\mathrm{End}_{\mathcal{C}}(M)$-linear monomorphism, $\phi_M(\eta)$ is almost vanishing on $\mathrm{End}_{\overline{\mathcal{C}}}(M)$. By Theorem 2.2, η is an almost split sequence in $\mathcal{C}$. The proof of the proposition is completed.

The following statement generalizes the main results stated in [8]; see also [11]. Observe that we impose no finiteness assumption.

3.6. Corollary. *Let $\mathcal{A}$ have almost split sequences. If $\mathcal{C}$ is Krull-Schmidt and functorially finite in $\mathcal{A}$, then $\mathcal{C}$ has almost split sequences.*

Proof. Assume that $\mathcal{C}$ is Krull-Schmidt and functorially finite in $\mathcal{A}$. Let $Z \in \mathcal{C}$ be indecomposable but not Ext-projective. In particular, Z is not Ext-projective in $\mathcal{A}$, and hence there exists an almost split sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$. By the assumption and Lemma 3.2, X has a minimal right injectively stable $\mathcal{C}$-approximation $f : M \to X$. By Lemma 3.3(1), f is non-zero, and hence, M is Krull-Schmidt. By Proposition 3.5, $\mathcal{C}$ has a almost split sequence ending with Z. This shows that $\mathcal{C}$ has right almost split sequences. Dually, $\mathcal{C}$ has left almost split sequences. The proof of the corollary is completed.

Next, we shall establish the converse of Proposition 3.5. For this purpose, some finiteness assumptions are required.

3.7. Proposition. *Let R be artinian, and let $\delta :\ 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ and $\eta :\ 0 \longrightarrow M \longrightarrow N \longrightarrow Z \longrightarrow 0$ be almost split sequences in $\mathcal{A}$ and in $\mathcal{C}$, respec-*

tively. If $\mathrm{Hom}_{\overline{\mathcal{C}}}(L, M) \in \mathrm{mod}R$ *for any* $L \in \mathcal{C}$*, then* η *embeds in a pushout diagram*

$$\begin{array}{rccccccccc} \eta: & 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & 0 \\ & & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ \delta: & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

in $\mathcal{A}$*; and in any such pushout diagram,* f *is a minimal right injectively stable* $\mathcal{C}$*-approximation of* X*, and* g *is a right injectively stable* $\mathcal{C}$*-approximation of* Y*.*

Proof. Suppose that $\mathrm{Hom}_{\overline{\mathcal{C}}}(L, M)$ is of finite R-length, for every $L \in \mathcal{C}$. Since δ is almost split, $\mathcal{A}$ has a commutative diagram

$$\begin{array}{rccccccccc} \eta: & 0 & \longrightarrow & M & \xrightarrow{r} & N & \xrightarrow{s} & Z & \longrightarrow & 0 \\ & & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \| & & \\ \delta: & 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0. \end{array}$$

Fix an R-linear form $\varphi : \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \to I$ such that $\varphi(\delta) \neq 0$. This yields an R-linear form

$$\psi : \mathrm{Ext}^1_{\mathcal{C}}(Z, M) \to I : \zeta \mapsto \varphi(f\zeta)$$

such that $\psi(\eta) = \varphi(\delta) \neq 0$. Let $L \in \mathcal{C}$. By Proposition 2.1, we have non-degenerate R-bilinear forms

$$<\,,\,>_{\varphi}\colon\ \mathrm{Hom}_{\overline{\mathcal{A}}}(L, X) \times \mathrm{Ext}^1_{\mathcal{A}}(Z, L) \to I : (\bar{q}, \zeta) \mapsto \varphi(q\zeta)$$

and

$$<\,,\,>_{\psi}\colon\ \mathrm{Hom}_{\overline{\mathcal{C}}}(L, M) \times \mathrm{Ext}^1_{\mathcal{C}}(Z, L) \to I : (\tilde{p}\,, \zeta) \mapsto \psi(p\zeta).$$

Let $q : L \to X$ be a morphism in $\mathcal{A}$. Since $\mathrm{Ext}^1_{\mathcal{A}}(Z, L) = \mathrm{Ext}^1_{\mathcal{C}}(Z, L)$, we have an R-linear form

$$<\bar{q},\, ->_{\varphi}\colon\ \mathrm{Ext}^1_{\mathcal{C}}(Z, L) \to I : \zeta \mapsto <\bar{q},\, \zeta>_{\varphi}\ .$$

Since $\mathrm{Hom}_{\overline{\mathcal{C}}}(L, M)$ is of finite R-length, by Lemma 2.4, there exists $p : L \to M$ in $\mathcal{C}$ such that $<\bar{q},\, ->_{\varphi} = <\tilde{p}\,,\, ->_{\psi}$. That is, for any $\zeta \in \mathrm{Ext}^1_{\mathcal{A}}(Z, L)$, we have

$$<\bar{q},\, \zeta>_{\varphi} = <\tilde{p}\,,\, \zeta>_{\psi} = \psi(p\,\zeta) = \varphi(f p\,\zeta) = <\overline{fp}\,, \zeta>_{\varphi}\ .$$

Since $<\,,\,>_{\varphi}$ is non-degenerate, $\bar{q} = \overline{fp}\,$. This shows that f is a right injectively stable $\mathcal{C}$-approximation of X, which is minimal since M is strongly indecomposable in both $\mathcal{A}$ and $\overline{\mathcal{A}}$; see (1.2).

Next, consider a morphism $h : L \to Y$ in $\mathcal{A}$ with $L \in \mathcal{C}$. Since vh is not a retraction, there exists $w : L \to N$ such that $vh = sw = vgw$. Thus $h - gw = ut$ for some $t : L \to X$. Using what we have just proved, there exists some morphism $j : L \to M$ such that $\bar{t} = \overline{fj}$, and hence $\overline{h - gw} = \overline{ufj} = \overline{grj}$. This yields that $\overline{h} = \bar{g}(\overline{w} + \overline{rj})$. That is, g is a right injectively stable $\mathcal{C}$-approximation of Y. The proof of the proposition is completed.

We are ready to obtain the main result of this section.

3.8. Theorem. *Let* $\mathcal{A}$ *be an exact* R*-category where* R *is artinian, and let* $\mathcal{C}$ *be an exact subcategory of* $\mathcal{A}$ *which is Ext-finite and Krull-Schmidt.*

(1) *If* $\mathcal{A}$ *has right almost split sequences, then* $\mathcal{C}$ *has right almost split sequences if and only if* $\tau_{\mathcal{A}}Z$ *has a right injectively stable* $\mathcal{C}$*-approximation, for any indecomposable not Ext-projective object* Z *in* $\mathcal{C}$*.*

(2) *If $\mathcal{A}$ has left almost split sequences, then $\mathcal{C}$ has left almost split sequences if and only if $\tau_{\mathcal{A}}^{-}X$ has a left projectively stable $\mathcal{C}$-approximation, for any indecomposable not Ext-injective object X in $\mathcal{C}$.*

Proof. We shall prove only Statement (1). Assume that $\mathcal{A}$ has right almost split sequences. As seen in the proof of Corollary 3.6, the sufficiency follows from Proposition 3.5 and Lemmas 3.2 and 3.3. Suppose now that $\mathcal{C}$ has right almost split sequences. Let $Z \in \mathcal{C}$ be indecomposable and not Ext-projective. Then $\mathcal{C}$ has an almost split sequence $0 \longrightarrow M \longrightarrow N \longrightarrow Z \longrightarrow 0$. For any $L \in \mathcal{C}$, since $\mathrm{Ext}^1_{\mathcal{C}}(Z, L)$ is of finite R-length, by Lemma 2.5(1), so is $\mathrm{Hom}_{\overline{\mathcal{C}}}(L, M)$. Thus, by Proposition 3.7, $\tau_{\mathcal{A}} Z$ has a minimal right injectively stable $\mathcal{C}$-approximation. The proof of the theorem is completed.

We conclude this section with an application. First, following [10], we shall say that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a *torsion theory* if the following two conditions are satisfied:

(1) For any objects $T \in \mathcal{T}$ and $F \in \mathcal{F}$, we have $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$.

(2) For any object $X \in \mathcal{A}$, there exists a canonical short exact sequence

$$0 \longrightarrow t(X) \longrightarrow X \longrightarrow f(X) \longrightarrow 0,$$

where $t(X) \in \mathcal{T}$ and $f(X) \in \mathcal{F}$.

Remark. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{A}$, then it is easy to see that $\mathcal{T}$ and $\mathcal{F}$ are exact subcategories of $\mathcal{A}$.

The following statement generalizes some of Hoshino's results formulated for modules over an artin algebra; see [13, Section 1]. Note that we do not impose any finiteness conditions.

3.9. Proposition. *Let $\mathcal{A}$ be an exact R-category with a torsion theory $(\mathcal{T}, \mathcal{F})$, and let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an almost split sequence in $\mathcal{A}$.*

(1) *If $Z \in \mathcal{T}$ is not Ext-projective, then $\mathcal{T}$ has an induced almost split sequence $0 \longrightarrow t(X) \longrightarrow t(Y) \longrightarrow Z \longrightarrow 0$.*

(2) *If $X \in \mathcal{F}$ is not Ext-injective, then $\mathcal{F}$ has an induced almost split sequence $0 \longrightarrow X \longrightarrow f(Y) \longrightarrow f(Z) \longrightarrow 0$.*

Proof. We shall only prove Statement (1). Suppose that $Z \in \mathcal{T}$ is not Ext-projective. Consider the canonical short exact sequence

$$0 \longrightarrow t(X) \xrightarrow{q} X \xrightarrow{p} f(X) \longrightarrow 0.$$

Observe that q is a right $\mathcal{T}$-approximation, and hence a right injectively stable $\mathcal{T}$-approximation, of X. By Lemma 3.3(1), q is not injectively trivial in $\mathcal{A}$. In particular, $t(X)$ is not Ext-injective in $\mathcal{A}$. Moreover, since q is a monomorphism, we have an R-linear isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(t(X), q) : \mathrm{End}_{\mathcal{A}}(t(X)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(t(X), X).$$

On the other hand, $\mathrm{Hom}_{\underline{\mathcal{A}}}(Z, f(X)) = 0$ since $Z \in \mathcal{T}$, and thus, $\mathrm{Ext}^1_{\mathcal{A}}(f(X), X) = 0$ by Proposition 2.1. Applying $\mathrm{Hom}_{\mathcal{A}}(-, X)$ to the canonical short exact sequence yields an R-linear epimorphism

$$\mathrm{Hom}_{\mathcal{A}}(q, X) : \mathrm{End}_{\mathcal{A}}(X) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(t(X), X).$$

Composing this with the inverse of $\mathrm{Hom}_{\mathcal{A}}(t(X), q)$, we get an R-linear epimorphism $\varphi : \mathrm{End}_{\mathcal{A}}(X) \to \mathrm{End}_{\mathcal{A}}(t(X))$, which is a ring morphism. Since $\mathrm{End}_{\mathcal{A}}(X)$ is local, so is $\mathrm{End}_{\mathcal{A}}(t(X))$. Hence, $q : t(X) \to X$ is a minimal right injectively stable $\mathcal{T}$-approximation of X. By Proposition 3.5, $\mathcal{A}$ has a pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t(X) & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0, \end{array}$$

where the upper row is an almost split sequence in $\mathcal{T}$. Using the Snake Lemma, we infer that $N \cong t(Y)$. The proof of the proposition is completed.

The following statement is an immediate consequence of the preceding result.

3.10. COROLLARY. *Let $\mathcal{A}$ be an exact R-category with a torsion theory $(\mathcal{T}, \mathcal{F})$.*
(1) *If $\mathcal{A}$ has right almost split sequences, then $\mathcal{T}$ has right almost split sequences.*
(2) *If $\mathcal{A}$ has left almost split sequences, then $\mathcal{F}$ has left almost split sequences.*

4. REPRESENTATION CATEGORIES OF INFINITE QUIVERS

The purpose of this section is to illustrate some results obtained in the previous sections. Let k a field, and let Q be an infinite strongly locally finite quiver. The category $\mathrm{rep}(Q)$ of locally finite dimensional k-representations of Q is hereditary, abelian, but not Hom-finite. To a vertex $x \in Q$, one associates an indecomposable projective representation P_x and an indecomposable injective representation I_x of Q. As in the finite case, we have a Nakayama equivalence

$$\nu : \mathrm{proj}(Q) \xrightarrow{\sim} \mathrm{inj}(Q),$$

where $\mathrm{proj}(Q)$ and $\mathrm{inj}(Q)$ are the full additive subcategories of $\mathrm{rep}(Q)$ generated by the P_x and by the I_x, respectively. Furthermore, let $\mathrm{rep}^+(Q)$ be the full subcategory of $\mathrm{rep}(Q)$ generated by the representations with a projective presentation lying in $\mathrm{proj}(Q)$, and let $\mathrm{rep}^-(Q)$ be the one generated by the representations with an injective co-presentation lying in $\mathrm{inj}(Q)$. If $M \in \mathrm{rep}^+(Q)$ is indecomposable and non-projective with a minimal projective presentation

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0,$$

then the kernel of $\nu(f)$, written as $\mathrm{DTr}\,M$, is an indecomposable non-injective object in $\mathrm{rep}^-(Q)$ such that $\mathrm{Hom}(L, \mathrm{DTr}\,M)$ is finite dimensional for every object $L \in \mathrm{rep}(Q)$. Dually, if $M \in \mathrm{rep}^-(Q)$ is indecomposable and non-injective with a minimal injective co-presentation

$$0 \longrightarrow M \longrightarrow I_0 \xrightarrow{g} I_1 \longrightarrow 0,$$

then the co-kernel of $\nu^-(g)$, written as $\mathrm{TrD}\,M$, is an indecomposable non-projective object in $\mathrm{rep}^+(Q)$ such that $\mathrm{Hom}(\mathrm{TrD}\,M, L)$ is finite dimensional for every object $L \in \mathrm{rep}(Q)$. In [9, (2.8)], by showing that $\mathrm{Ext}^1(M, -) \cong D\mathrm{Hom}(-, \mathrm{DTr}\,M)$ and $\mathrm{Ext}^1(-, M) \cong D\mathrm{Hom}(\mathrm{TrD}\,M, -)$, one obtained the following result; compare 2.6.

4.1. THEOREM. *Let M be an indecomposable representation in* $\mathrm{rep}(Q)$.
(1) *If $M \in \mathrm{rep}^+(Q)$ is not projective, then* $\mathrm{rep}(Q)$ *has an almost split sequence*
$0 \longrightarrow \mathrm{DTr}\,M \longrightarrow E \longrightarrow M \longrightarrow 0.$

(2) *If* $M \in \operatorname{rep}^-(Q)$ *is not injective, then* $\operatorname{rep}(Q)$ *has an almost split sequence* $0 \longrightarrow N \longrightarrow F \longrightarrow \operatorname{TrD} N \longrightarrow 0$.

Now, it has been shown that $\operatorname{rep}^+(Q)$ and $\operatorname{rep}^-(Q)$ are exact subcategories of $\operatorname{rep}(Q)$ which are Hom-finite, hereditary, abelian, and have as intersection the category of finite dimensional representations; see [9, (1.15)]. As another application of the results of the previous section, we give a short proof of the following result which is obtained in [9, Section 3].

4.2. THEOREM. *Let* M *be an indecomposable representation in* $\operatorname{rep}(Q)$.

(1) *If* $M \in \operatorname{rep}^+(Q)$ *is not projective, then* $\operatorname{rep}^+(Q)$ *has an almost split sequence* $0 \longrightarrow L \longrightarrow N \longrightarrow M \longrightarrow 0$ *if and only if* $\operatorname{DTr} M$ *is finite dimensional*; *and in this case,* $L \cong \operatorname{DTr} M$.
(2) *If* $M \in \operatorname{rep}^-(Q)$ *is not injective, then* $\operatorname{rep}^-(Q)$ *has an almost split sequence* $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ *if and only if* $\operatorname{TrD} M$ *is finite dimensional*; *and in this case,* $L \cong \operatorname{TrD} M$.

Proof. We prove only Statement (1). Let $M \in \operatorname{rep}^+(Q)$ be not projective. Then, $\operatorname{rep}(Q)$ has an almost split sequence $\delta:\ 0 \longrightarrow \operatorname{DTr} M \longrightarrow N \longrightarrow M \longrightarrow 0$, where $\operatorname{DTr} M \in \operatorname{rep}^-(Q)$; see [9, (2.8)]. If $\operatorname{DTr} M$ is finite dimensional, then δ lies in $\operatorname{rep}^+(Q)$, and hence it is an almost split sequence in $\operatorname{rep}^+(Q)$.

Conversely, assume that $0 \longrightarrow L \longrightarrow N \longrightarrow M \longrightarrow 0$ is an almost split sequence in $\operatorname{rep}^+(Q)$. By Proposition 3.7, $\operatorname{DTr} M$ has a right injectively stable $\operatorname{rep}^+(Q)$-approximation $f : L \to \operatorname{DTr} M$. Suppose that $\operatorname{DTr} M$ is infinite dimensional. Then $\operatorname{supp}(\operatorname{DTr} M)$ contains a left infinite path; see [9, (1.7)], which does not lie in $\operatorname{supp} L$; see [9, (1.6)]. In particular, there exists a vertex x in $\operatorname{supp}(\operatorname{DTr} M)$ which does not lie in $\operatorname{supp} L$. Then $\operatorname{Hom}_{\operatorname{rep}(Q)}(P_x, L) = 0$, but there exists a non-zero morphism $g : P_x \to \operatorname{DTr} M$. Write $g = jh$, where $j : X \to \operatorname{DTr} M$ is a monomorphism and $h : P_x \to X$ is an epimorphism. Then, we have $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, L) = 0$. Moreover, $\operatorname{supp} X$ is contained in the intersection of $\operatorname{supp} P_x$ and $\operatorname{supp}(\operatorname{DTr} M)$. Note that $\operatorname{supp}(\operatorname{DTr} M)$ has some vertices $a_1, \ldots, a_r$ such that every vertex in $\operatorname{supp}(\operatorname{DTr} M)$ is a predecessor of some of the a_i; see [9, (1.6)]. Since Q is interval-finite, $\operatorname{supp} X$ is finite. Thus X is finite dimensional. In particular, X has an injective envelope $q : X \to J$ with $J \in \operatorname{inj}(Q)$. Observe that $\bar{j}$ factors through $\bar{f}$. Then $\bar{j} = \bar{0}$, since $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, L) = 0$. That is, j is injectively trivial, and hence j factors through q. On the other hand, since $\operatorname{rep}(Q)$ is hereditary and $\operatorname{DTr} M$ is indecomposable, $\operatorname{Hom}_{\operatorname{rep}(Q)}(J,\ \operatorname{DTr} M) = 0$. This yields $j = 0$, a contradiction. The proof of the theorem is completed.

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