Some Homological Conjectures for Quasi-stratified Algebras

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Introduction

In this paper, we are mainly concerned with the *Cartan determinant conjecture* and the *no loop conjecture*. If A is an artin algebra of finite global dimension, the first conjecture claims that the Cartan determinant of A is equal to 1, while the second one states that every simple A-module admits only the trivial self-extension. Among numerous partial solutions to these conjectures such as those in [6, 14, 15, 22, 24], we observe particularly that both of them have been established for standardly stratified algebras; see [4, 21]. This class of algebras serves as a generalization of quasi-hereditary algebras introduced by Cline, Parshall and Scott; see, for example, [8]. The key idea for studying standardly stratified algebras is to relate the homological properties of an algebra A to those of A/I with I an idempotent projective ideal. We shall pursue further in this line by relaxing the condition that I be idempotent. This enables us to generalize many results found in [4, 11, 21, 24]. More importantly, it leads us to the introduction of two new classes of algebras, called *quasi-stratified* and *ultimate-hereditary* algebras, which include standardly stratified and quasi-hereditary algebras, respectively. We shall show that the finiteness of the global dimension of a quasi-stratified algebra is equivalent to the Cartan determinant equal to one, as well as to the algebra being ultimate-hereditary. Moreover, in this case, we prove that every simple module admits only the trivial self-extension.

Remarkably, the no loop conjecture has been verified for finite dimensional algebras over a field given by quivers with relations; see [18, 19]. A stronger version, called the *strong no loop conjecture*, states that every simple module of finite projective dimension over an artin algebra admits only the trivial self-extension. This remains open except for algebras which are monomial [18] or special biserial [20]. We refer to [7, 16] for more special cases. The last result of this paper is to confirm the strong no loop conjecture for algebras which are quasi-stratified on one side, and in particular, for standardly stratified algebras.

1. Projective ideals and quasi-stratifications

Throughout this paper, A stands for an artin algebra. The radical and the global dimension of A will be written as rad$A$ and gdim(A), respectively. The category of finitely generated right A-modules and that of finitely generated left A-modules will be denoted by mod-A and A-mod, respectively. Moreover, D stands for the usual duality between these categories.

Let I be an ideal (that is, a two-sided ideal) of A. We say that I is *right* (respectively, *left*) *projective* if the right A-module I_A (respectively, the left A-module ${}_AI$) is projective. For brevity, we say that I is *projective* if it is either right or left projective. Furthermore, let t be the smallest positive integer such that $I^t = I^{t+1}$. Then I^t is the maximal idempotent ideal of A contained in I. We shall call t and I^t the *idempotency* and the *idempotent part* of I, respectively. In this case, it is well known that I^t is generated by an idempotent; see, for example, [11, Statement 6]. The main objective of this section is to relate the homological properties of A to those of A/I with I being projective. Let us start with an easy observation.

1.1. Lemma. *Let I be an ideal of A with idempotent part J. If I is right projective, then J is an idempotent right projective ideal of A, while I/J is a nilpotent right projective ideal of A/J.*

Proof. Assume that I_A is projective of idempotency t. Then $J = I^t$ is projective as a right A-module. Since $IJ = I^{t+1} = I^t = J$, $I/J = I/IJ$ is projective as a right A/J-module. This completes the proof of the lemma.

For a module M in mod-A, we write $\mathrm{pdim}_A(M)$ for the projective dimension of M over A. The following result is essential for our investigation.

1.2. Lemma. *Let I be a right projective ideal of A of idempotency t. For every module M in* mod-A/I*, we have*
(1) $\mathrm{pdim}_A(M) \le \mathrm{pdim}_{A/I}(M) + 1$, *and*
(2) $\mathrm{pdim}_{A/I}(M) \le \mathrm{pdim}_A(M) + 2(t-1)$.

Proof. The statement (1) is well-known; see, for example, [11, Statement 1]. In order to prove (2), write $B = A/I$ and let M be a module in mod-B. Clearly, we need only to consider the case where $\mathrm{pdim}_A(M) = r < \infty$. If $r = 0$, then M is a projective A-module annihilated by I. Hence $M = M/MI$ is projective over B. This proves (2) for $r = 0$. If $r = 1$, then mod-A has a short exact sequence

$$0 \longrightarrow Q \xrightarrow{j} P \longrightarrow M \longrightarrow 0$$

with j an inclusion map between projective modules. Since $MI = 0$, we get $PI \subseteq Q$, and hence a chain

$$PI^{t+1} \subseteq QI^t \subseteq PI^t \subseteq QI^{t-1} \subseteq PI^{t-1} \subseteq \cdots \subseteq QI \subseteq PI \subseteq Q \subseteq P$$

of submodules of P. This gives rise to an exact sequence

$$(*) \quad PI^t/QI^t \to QI^{t-1}/QI^t \to PI^{t-1}/PI^t \to \cdots \to Q/QI \to P/PI \to M \to 0$$

in mod-B. Since I_A is a projective A-module, so are the PI^i and the QI^i. As a consequence, the PI^i/PI^{i+1} and the QI^i/QI^{i+1} are projective modules in mod-B. Moreover, $PI^t/QI^t = 0$ since $PI^t = PI^{t+1} \subseteq QI^t \subseteq PI^t$. Thus $(*)$ is a projective resolution of M over B. In particular, $\mathrm{pdim}_B(M) \leq 2(t-1)+1$. This proves that (2) holds for $r = 1$. Assume now that $\mathrm{pdim}_A(M) = r > 1$ and that (2) holds for modules N in mod-B with $\mathrm{pdim}_A(N) \leq r-1$. Consider a short exact sequence

$$0 \longrightarrow \Omega \xrightarrow{\ j\ } P \xrightarrow{\ \varepsilon\ } M \longrightarrow 0$$

in mod-A with P projective and j an inclusion map. Then $\mathrm{pdim}_A(\Omega) = r-1$, and there exists a short exact sequence

$$0 \longrightarrow \Omega/PI \longrightarrow P/PI \longrightarrow M \longrightarrow 0$$

in mod-B with P/PI projective. In particular, $\mathrm{pdim}_B(M) \leq \mathrm{pdim}_B(\Omega/PI)+1$. Now the projectivity of PI implies that $\mathrm{pdim}_A(\Omega/PI) \leq \mathrm{pdim}_A(\Omega) = r-1$. By the induction hypothesis, $\mathrm{pdim}_B(\Omega/PI) \leq r-1+2(t-1)$. Therefore,

$$\mathrm{pdim}_B(M) \leq \mathrm{pdim}_B(\Omega/PI)+1 \leq r-1+2(t-1)+1 = \mathrm{pdim}_A(M)+2(t-1).$$

This completes the proof of the lemma.

For convenience, we define $\mathrm{gdim}(0) = -1$. The following result generalizes Statement 4 in [11].

1.3. PROPOSITION. *Let I be a projective ideal of A of idempotency t, and let e be an idempotent which generates the idempotent part of I. Then*

(1) $\mathrm{gdim}(A/I) \leq \mathrm{gdim}(A) + 2(t-1)$.

(2) $\mathrm{gdim}(eAe) \leq \mathrm{gdim}(A) \leq \mathrm{gdim}(eAe) + \mathrm{gdim}(A/I) + 2$.

Proof. Assume that I is right projective. The statement (1) follows immediately from Lemma 1.2(2). We shall now prove the first inequality in (2). For this purpose, we may assume that $e \neq 0$. By Lemma 1.1, AeA_A is projective, and consequently, AeA_A lies in $\mathrm{add}(eA)$, the full subcategory of mod-A generated by the direct sums of the direct summands of eA. Thus $\mathrm{Hom}_A(eA, AeA)$ is a projective right module over $\mathrm{End}_A(eA)$; see, for example [2, (II.2.1)], that is, Ae is projective in mod-eAe. It then follows easily that Pe is projective in mod-eAe whenever P is a projective module in mod-A. Let S be a simple right A-module such that $Se \neq 0$. If

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to S \to 0$$

is a finite projective resolution of S over A, then

$$0 \to P_ne \to P_{n-1}e \to \cdots \to P_1e \to P_0e \to Se \to 0$$

is a finite projective resolution of Se over eAe. Thus $\mathrm{gdim}(eAe) \leq \mathrm{gdim}(A)$.

In order to show the second inequality in (2), we need only to consider the case where $\mathrm{gdim}(eAe) = r < \infty$ and $\mathrm{gdim}(A/I) = s < \infty$. We begin with the following claim: if N is in mod-A such that $NI^m = 0$ for some $m > 0$, then $\mathrm{pdim}_A(N) \le s + 1$. Indeed, if $m = 1$, then N is a module over A/I. Hence by Lemma 1.2(1), $\mathrm{pdim}_A(N) \le \mathrm{pdim}_{A/I}(N) + 1 \le s + 1$. In particular, $\mathrm{pdim}_A(M/MI) \le s + 1$ for all $M \in$ mod-A. Suppose now that $m > 1$. Since $(NI)I^{m-1} = 0$, by the induction hypothesis, $\mathrm{pdim}_A(NI) \le s + 1$. This gives rise to $\mathrm{pdim}_A(N) \le \max\{\mathrm{pdim}_A(NI), \mathrm{pdim}_A(N/NI)\} \le s + 1$. Our claim is proved.

If $e = 0$, then $I^t = 0$ and $\mathrm{gdim}(eAe) = -1$. It then follows from our claim that $\mathrm{gdim}(A) \le \mathrm{gdim}(A/I) + 1 = \mathrm{gdim}(eAe) + \mathrm{gdim}(A/I) + 2$. Assume now that $e \neq 0$. Let M be a module in mod-A, and let Ω_r be the r-th syzygy of M. Then mod-A admits an exact sequence

$$0 \to \Omega_r \to Q_{r-1} \to \cdots \to Q_1 \to Q_0 \to M \to 0$$

with the Q_i projective, which induces an exact sequence

$$0 \to \Omega_r e \to Q_{r-1}e \to \cdots \to Q_1 e \to Q_0 e \to Me \to 0$$

in mod-eAe with the $Q_i e$ projective. Since $\mathrm{pdim}_{eAe}(Me) \le r$, we see that $\Omega_r e$ is eAe-projective. Consider now a short exact sequence

$$0 \longrightarrow L \xrightarrow{\ j\ } P \xrightarrow{\ \varepsilon\ } \Omega_r eA \longrightarrow 0$$

in mod-A with j an inclusion and ε a projective cover of $\Omega_r eA$. It induces a short exact sequence

$$0 \longrightarrow Le \longrightarrow Pe \xrightarrow{\ \bar{\varepsilon}\ } \Omega_r e \longrightarrow 0$$

in mod-eAe with Pe projective. Noting that P lies in $\mathrm{add}(eA)$, we see that $(\mathrm{rad}P)e$ is contained in the radical of the eAe-module Pe. Thus $\bar{\varepsilon}$ is a projective cover of $\Omega_r e$ in mod-eAe. Now the projectivity of $\Omega_r e$ implies that $Le = 0$, that is, $LI^t = 0$. It follows from the above claim that $\mathrm{pdim}_A(L) \le s + 1$, and hence $\mathrm{pdim}_A(\Omega_r eA) \le s + 2$. For the same reason, we have $\mathrm{pdim}_A(\Omega_r/\Omega_r eA) \le s + 1$. Therefore, $\mathrm{pdim}_A(\Omega_r) \le \max\{\mathrm{pdim}_A(\Omega_r eA),\ \mathrm{pdim}_A(\Omega_r/\Omega_r eA)\} \le s + 2$. This gives rise to $\mathrm{pdim}_A(M) \le r + \mathrm{pdim}_A(\Omega_r) \le r + s + 2$. The proof of the proposition is completed.

As an immediate consequence, we have the following interesting result.

1.4. COROLLARY. *Let I be a projective ideal of A, and let e be an idempotent which generates the idempotent part of I. Then A is of finite global dimension if and only if eAe and A/I are of finite global dimension.*

Before proceeding further, we need some terminology on idempotents. Let e be an idempotent of A. We say that e is *simple* if e is primitive such that

$e\,\mathrm{rad}A\,e = 0$, or equivalently, eAe is a simple artin algebra. For convenience, we say that e is *pseudo-primitive* if e is zero or primitive, and *pseudo-simple* if e is zero or simple.

1.5. Definition. (1) An ideal of A is called *right* (respectively, *left*) *quasi-stratifying* if it is right (respectively, left) projective and its idempotent part is generated by a pseudo-primitive idempotent.

(2) A right (respectively, left) quasi-stratifying ideal of A is called *right* (respectively, *left*) *quasi-heredity* if its idempotent part is generated by a pseudo-simple idempotent.

If I is a right quasi-stratifying ideal of A and e is a pseudo-primitive idempotent which generates the idempotent part of I, then it is easy to see that I is right quasi-heredity if and only if $e\,\mathrm{rad}A\,e = 0$. For brevity, we say that an ideal of A is *quasi-stratifying* (respectively, *quasi-heredity*) if it is right or left quasi-stratifying (respectively, right or left quasi-heredity).

Recall that A is *right standardly stratified* (respectively, *quasi-hereditary*) if A admits a chain of ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = A$$

such that I_{i+1}/I_i is a right projective ideal of A/I_i generated by a primitive (respectively, simple) idempotent, for all $0 \le i < n$; see [10] for more equivalent conditions. Note that a right standardly stratified algebra is called a *QH-1 algebra* in [21].

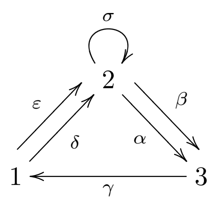
1.6. Definition. We call A *quasi-stratified* if A admits a chain of ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

such that I_{i+1}/I_i is a quasi-stratifying ideal of A/I_i, for all $0 \le i < r$. Such a chain is called a *quasi-stratification.*

Note that the notion of a quasi-stratified algebra is left-right symmetric. Moreover, standardly stratified algebras are clearly quasi-stratified.

1.7. Example. Let A be the algebra over a field given by the quiver



with relations $\sigma^2 = \sigma\beta = \beta\gamma = \gamma\delta = \varepsilon\alpha = \varepsilon\sigma = \varepsilon\beta = \delta\alpha - \delta\sigma\alpha = 0$.

It is easy to see that A is neither right nor left standardly stratified. However, one can verify that the chain

$$0 \subset <\varepsilon> \subset <\varepsilon,\alpha> \subset <\varepsilon,\alpha,\delta>_A \subset <\varepsilon,\alpha,\delta,e_2> \subset <\varepsilon,\alpha,\delta,e_2,e_3> \subset A$$

is quasi-stratification of A, where the projectivity for the first non-zero ideal is on the left, while that for other ideals is on the right. Thus A is quasi-stratified.

1.8. Definition. We call A *ultimate-hereditary* if A admits a chain of ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

such that I_{i+1}/I_i is a quasi-heredity ideal of A/I_i, for all $0 \le i < r$. Such a chain is called *quasi-heredity.*

It is clear that a quasi-hereditary algebra is ultimate-hereditary. Moreover, if I is a quasi-heredity ideal of A such that A/I is ultimate-hereditary, then A is ultimate-hereditary.

The Cartant determinant conjecture has been verified by Wilson for positively gradable algebras; see [22]. The referee has drawn our attention to the existence of a quasi-hereditary algebra which is not positively gradable. Based on this example, we shall construct an ultimate-hereditary algebra which is neither positively gradable nor quasi-hereditary.

1.9. Example. Let k be a field, and consider the k-algebra

$$C = k < X, Y > / < X^3, XY, YX^2, X^2 - Y^3, YX - Y^3 > .$$

Setting

$$M_C = C \oplus C/\mathrm{rad}(C) \oplus C/\mathrm{rad}^2(C) \oplus C/\mathrm{rad}^3(C),$$

one gets a quasi-hereditary algebra $B = \mathrm{End}_C(M)$, which is not positively gradable; see [3]. Since B is elementary, we may assume that $B = kQ_B/I_B$ with (Q_B, I_B) a bound quiver. Let a, b, c, d be the vertices of Q_B which correspond to the summands

$$C,\ C/\mathrm{rad}(C),\ C/\mathrm{rad}^2(C),\ C/\mathrm{rad}^3(C)$$

of M respectively. Note that b is neither a sink nor a source of Q_B. Moreover, among the canonical primitive idempotents of B, the one associated to b is the only simple idempotent. We now construct a new quiver Q from Q_B by adding a new vertex x and two new arrows $\alpha : b \to x$ and $\beta : x \to b$. Choose an arrow γ of Q_B which ends at b. We claim that $A = kQ/I$ with $I =< I_B, \alpha\beta, \gamma\alpha >$ is an ultimate-hereditary algebra which is neither positively gradable nor quasi-hereditary. Indeed, denote by e_a, e_b, e_c, e_d, e_x the primitive idempotent of A associated to a, b, c, d, x, respectively. Since $\alpha\beta \in I$, we have $B \cong eAe$ with $e = e_a + e_b + e_c + e_d$. If A admits a positive grading $A = \oplus_{i\ge 0} A_i$, then

$eAe = \oplus_{i\geq 0} eA_ie$ with $\mathrm{rad}(eAe) = e\,\mathrm{rad}(A)\,e = \oplus_{i\geq 1} eA_ie$ is a positive grading of eAe, which is contrary to the non-gradablity of B. Moreover, e_b is the only simple idempotent in $\{e_a, e_b, e_c, e_d, e_x\}$. In particular, $Ae_bAe_x = A\bar{\alpha}$, where $\bar{\alpha} = \alpha + I$. Since $\gamma\alpha \in I$, the left A-module $A\bar{\alpha}$ is not projective. Thus Ae_bA is not heredity. This shows that A is not quasi-hereditary. Finally, since $\alpha\beta \in I$, we have $A\bar{\beta}A = \bar{\beta}A$. Since there exists no relation on Q starting with β, we have $\bar{\beta}A_A \cong A_A$. Thus $A\bar{\beta}A$ is right projective, and hence right quasi-heredity in A. Further, it is clear that $< \bar{\beta}, e_x > / < \bar{\beta} >$ is left projective, and hence left quasi-stratifying in $A/ < \bar{\beta} >$. Since $A/ < \bar{\beta}, e_x > \cong B$, we conclude that A is ultimate-hereditary.

For a module M in mod-A, $\ell\ell(M_A)$ denotes the Loewy length of M over A.

1.10. Lemma. *If A admits a quasi-stratification of length one, then A is Morita equivalent to eAe for every primitive idempotent e of A. Moreover, in this case, A is ultimate-hereditary if and only if A is hereditary.*

Proof. Assume that A is a quasi-stratifying ideal of itself. Being idempotent, $A = Ae_0A$ with e_0 a primitive idempotent. If e is an arbitrary primitive idempotent of A, it is easy to see that $eA \cong e_0A$ and $A = AeA$. This shows the first part of the lemma.

For the second part of the lemma. it suffices to show the necessity. For doing so, suppose that $A = Ae_0A$ with e_0 a primitive idempotent and A is ultimate-hereditary. Let I be a non-zero quasi-heredity ideal of A, and let e_1 be a pseudo-simple idempotent which generates the idempotent part of I. We consider only the case where I_A is projective. Then $I_A \cong (e_0A)^s$ for some $s > 0$, and hence $\ell\ell(I_A) = \ell\ell_A(e_0A_A) = \ell\ell(A_A)$. In particular, I is not nilpotent. Hence $e_1 \neq 0$, that is, e_1 is simple. Since A is Morita equivalent to e_1Ae_1, which is a simple algebra, A is hereditary. This completes the proof of the lemma.

We now give a bound on the global dimension of an ultimate-hereditary algebra in terms of the number of the non-isomorphic simple modules and the length of a quasi-heredity chain.

1.11. Proposition. *Let A be an ultimate-hereditary algebra with n non-isomorphic simple modules and a quasi-heredity chain of length r. Then*

$$\mathrm{gdim}(A) \leq \min\{2(r-1),\ n+r-2\}.$$

Proof. Let $0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$ be a quasi-heredity chain of A, and let e be a pseudo-simple idempotent which generates the idempotent part of I_1. We shall proceed by induction on r. If $r = 1$ then, by Lemma 1.10, $n = 1$ and $\mathrm{gdim}(A) = 0$. Assume now that $r > 1$. Then A/I_1 is ultimate-hereditary with a quasi-heredity chain

$$0 = I_1/I_1 \subset \cdots \subset I_{r-1}/I_1 \subset I_r/I_1 = A/I_1.$$

If $e = 0$, then A/I_1 has n non-isomorphic simple modules. By the induction hypothesis, $\operatorname{gdim}(A/I_1) \le \min\{2(r-2),\ n+r-3\}$. In view of Proposition 1.3(2), we see that $\operatorname{gdim}(A) \le \operatorname{gdim}(A/I_1)+1 \le \min\{2(r-1),\ n+r-2\}$. Suppose now that $e \ne 0$. Then e is primitive such that $\operatorname{gdim}(eAe) = 0$. Note that the number of non-isomorphic simple A/I_1-modules is $n-1$. By the induction hypothesis, $\operatorname{gdim}(A/I_1) \le \min\{2(r-2), (n-1)+(r-1)-2\}$. Applying Proposition 1.3(2), we get $\operatorname{gdim}(A) \le \operatorname{gdim}(A/I_1) + 2 \le \min\{2(r-1), n+r-2\}$. The proof of the proposition is completed.

Note that the length of a heredity chain of an artin algebra is bound by the number of the non-isomorphic simple modules. In this way, we recover a result of Dlab and Ringel saying that the global dimension of a quasi-hereditary algebra of n non-isomorphic simple modules is at most $2(n-1)$; see [11].

2. The Cartan determinant

The objective of this section is to study the Cartan determinant of a quasi-stratified algebra. We begin with a brief recall. Let $\{e_1, \ldots, e_n\}$ be a basic set of primitive idempotents of A, that is, $e_1A, \ldots, e_nA$ are the non-isomorphic indecomposable projective modules in mod-A. For $1 \le i, j \le n$, let c_{ij} be the multiplicity of the simple module $e_iA/e_i \operatorname{rad}A$ as a composition factor of e_jA. Then $(c_{ij})_{n\times n}$ is called a *right Cartan matrix* of A. Similarly, $\{e_1, \ldots, e_n\}$ determines a *left Cartan matrix* of A. Since A is an artin algebra, the right Cartan matrices and the left Cartan matrices of A all have the same determinant; see, for example, [13, (1.2)], which is called the *Cartan determinant* of A and denoted by $\operatorname{cd}(A)$. A well-known result of Eilenberg's, which is the origin of the Cartan determinant conjecture, says that $\operatorname{cd}(A) = \pm 1$ if A is of finite global dimension; see [12].

We first relate $\operatorname{cd}(A)$ and $\operatorname{cd}(A/I)$ with I a projective ideal. The following proposition generalizes the results stated in [21, (1.4)] and [4, (1.3)]. We refer to [5, 17, 23] for more similar matrix reductions. For convenience, we define $\operatorname{cd}(0) = 1$.

2.1. Proposition. *Let I be a projective ideal of A, and let e be an idempotent which generates the idempotent part of I. Then*

$$\operatorname{cd}(A) = \operatorname{cd}(eAe)\operatorname{cd}(A/I).$$

Proof. Since the right Cartan matrices and the left Cartan matrices of A have the same determinant, we need only to consider the case where I is right projective such that $B = A/I$ is nonzero. For $x \in A$, we write $\bar{x} = x + I \in B$. Let $\{e_1, \ldots, e_n\}$ be a basic set of primitive idempotents of A. For a module M in mod-A, we denote by $c_i(M)$ the multiplicity of $e_iA/e_i \operatorname{rad}A$ as a composition factor of M. Then $C(A) = (c_i(e_jA))_{n\times n}$ is a right Cartan matrix of A. For every $e_i \notin I$, we have $e_iA/e_i\,\mathrm{rad}A \cong \bar{e}_iB/\bar{e}_i\,\mathrm{rad}B$ as A-modules. As a consequence, for every B-module N, the multiplicity $d_i(N)$ of $\bar{e}_iB/\bar{e}_i\,\mathrm{rad}B$ as a composition factor of N coincides with $c_i(N)$. Moreover, the nonzero classes of $\bar{e}_1, \ldots, \bar{e}_n$ form a basic set of primitive idempotents of B.

(1) Suppose first that $e = 0$, that is, $I \subseteq \mathrm{rad}A$. Then $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basic set of primitive idempotents of B and $C(B) = (d_i(\bar{e}_jB))_{n\times n}$ is a right Cartan matrix of B. We may assume, without loss of generality, that

$$\ell\ell(e_1A_A) \leq \ell\ell(e_2A_A) \leq \cdots \leq \ell\ell(e_nA_A).$$

Since e_jI is projective with $\ell\ell(e_jI_A) < \ell\ell(e_jA_A)$, we have $e_1I = 0$ and

$$e_jI \cong (e_1A)^{r_{j1}} \oplus \cdots \oplus (e_{j-1}A)^{r_{j,j-1}},\ r_{ji} \geq 0,\ j = 2, \ldots, n.$$

Since $\bar{e}_jB \cong e_jA/e_jI$ as right A-modules, we deduce that $d_i(\bar{e}_1B) = c_i(e_1A)$ for $i = 1, \ldots, n$, and

$$d_i(\bar{e}_jB) = c_i(e_jA) - \sum_{s=1}^{j-1} r_{js}c_i(e_sA), \quad i = 1, \ldots, n;\ j = 2, \ldots, n.$$

This shows that the first column of $C(B)$ coincides with that of $C(A)$. More importantly, $C(B)$ can be obtained from $C(A)$ by some elementary column operations. As a consequence, $\det C(A) = \det C(B)$, that is, $\mathrm{cd}(A) = \mathrm{cd}(eAe)\,\mathrm{cd}(B)$ since $\mathrm{cd}(eAe) = 1$ in this case.

(2) Suppose now that $I = AeA$ is nonzero. We may assume, without loss of generality, that $\{e_1, \ldots, e_m\}$ with $1 \leq m < n$ is a basic set of primitive idempotents of eAe. It is easy to see that $I = A(e_1 + \cdots + e_m)A$ and that $C(eAe) = (c_i(e_jA))_{m\times m}$ is a right Cartan matrix of eAe. Now $\{\bar{e}_{m+1}, \ldots, \bar{e}_n\}$ is a basic set of primitive idempotents of B, and $C(B) = (d_i(\bar{e}_jB))_{m<i,j\leq n}$ is a right Cartan matrix of B. Fix an integer j with $m < j \leq n$. Since $e_jI = e_jA(e_1 + \cdots + e_m)A$, we have $e_jI \cong (e_1A)^{t_{j1}} \oplus \cdots \oplus (e_mA)^{t_{jm}}, \quad t_{ji} \geq 0$. Therefore,

$$c_i(e_jA/e_jI) = c_i(e_jA) - \sum_{s=1}^{m} t_{js}c_i(e_sA),\ i = 1, \ldots, n.$$

Since $c_i(e_jA/e_jI) = 0$ for $1 \leq i \leq m$, we get

$$c_i(e_jA) - \sum_{s=1}^{m} t_{js}c_i(e_sA) = 0, \quad i = 1, \ldots, m,$$

and

$$c_i(e_jA) - \sum_{s=1}^{m} t_{js}c_i(e_sA) = d_i(\bar{e}_jB), \quad i = m+1, \ldots, n.$$

This shows that $C(A)$ can be reduced by some elementary column operations to a matrix of the form

$$\begin{pmatrix} C(eAe) & 0 \\ * & C(B) \end{pmatrix}.$$

As a consequence, $\det C(A) = \det C(eAe) \det C(B)$.

(3) In general, by Lemma 1.1, AeA is a right projective ideal of A and I/AeA is a nilpotent right projective ideal of A/AeA such that $(A/AeA)/(I/AeA) \cong B$. Thus $\mathrm{cd}(A/AeA) = \mathrm{cd}(B)$ as shown in (1), and $\mathrm{cd}(A) = \mathrm{cd}(eAe)\,\mathrm{cd}(A/AeA)$ as seen in (2). This completes the proof of the proposition.

We shall now give two consequences of the above result. The first one generalizes some key results stated in [24].

2.2. COROLLARY. *Let e be an idempotent of A. If $e\,\mathrm{rad}A$ or $\mathrm{rad}A\,e$ is projective, then*

(1) $\mathrm{cd}(A) = \mathrm{cd}((1-e)A(1-e))$, *and*

(2) $\mathrm{gdim}((1-e)A(1-e)) \le \mathrm{gdim}(A) \le \mathrm{gdim}((1-e)A(1-e)) + 3$.

Proof. We need only to consider the case where e and $1-e$ are nonzero such that $e\,\mathrm{rad}A$ is projective. Write $e = e_1 + \cdots + e_r$, $1-e = e_{r+1} + \cdots + e_n$, where $e_1, \ldots, e_n$ are pairwise orthogonal primitive idempotents. If e_rA is isomorphic to a direct summand of $(1-e)A$, then $f = e_1 + \cdots + e_{r-1}$ is such that $f\mathrm{rad}A$ is projective and $(1-e)A(1-e)$ is Morita equivalent to $(1-f)A(1-f)$. Thus we may assume that none of the e_iA with $1 \le i \le r$ is isomorphic to a direct summand of $(1-e)A$. Then $eA(1-e)A = e(\mathrm{rad}A)\,(1-e)A$. We first claim that the ideal $A(1-e)A$ is right projective. That is, $e_iA(1-e)A$ is projective for all $1 \le i \le n$. This is evident for $r < i \le n$. It remains to show, for $1 \le i \le r$, that $e_iA(1-e)A$, or equivalently, $e_i(\mathrm{rad}A)(1-e)A$ is projective. For doing so, assume that

$$\ell\ell(e_1A_A) \ge \cdots \ge \ell\ell(e_{r-1}A_A) \ge \ell\ell(e_rA_A).$$

Since $e_r\,\mathrm{rad}A$ is projective with $\ell\ell(e_r\,\mathrm{rad}A_A) < \ell\ell(e_rA_A)$, it follows from the above inequalities that none of the e_iA with $1 \le i \le r$ is isomorphic to a direct summand of $e_r\,\mathrm{rad}A$. Thus $e_r\,\mathrm{rad}A \cong \bigoplus_{i=r+1}^n (e_iA)^{n_{ri}}$ with $n_{ri} \ge 0$. This gives rise to $e_r(\mathrm{rad}A)(1-e)A = e_r\mathrm{rad}A$, which is a projective module. Let s be an integer with $1 \le s < r$ such that the $e_i(\mathrm{rad}A)(1-e)A$ is projective for all $s < i \le r$. As we argued above, $e_s\mathrm{rad}A \cong \bigoplus_{i=s+1}^n (e_iA)^{n_{si}}$ with $n_{si} \ge 0$. Therefore,

$$e_s(\mathrm{rad}A)(1-e)A \cong \bigoplus_{i=s+1}^{n} (e_iA(1-e)A)^{n_{si}},$$

which is projective by the induction hypothesis. This proves our claim. Furthermore, we have $e\,\mathrm{rad}A \cap A(1-e)A \subseteq eA(1-e)A \subseteq e(\mathrm{rad}A)(1-e)A$, and hence $e\,\mathrm{rad}A \cap A(1-e)A = e\,\mathrm{rad}A \cdot A(1-e)A$. Therefore,

$$\begin{aligned} \mathrm{rad}\,(A/A(1-e)A) &= (e\,\mathrm{rad}A + A(1-e)A)/A(1-e)A \\ &\cong e\,\mathrm{rad}A/(e\,\mathrm{rad}A \cap A(1-e)A) \\ &\cong e\,\mathrm{rad}A/(e\,\mathrm{rad}A \cdot A(1-e)A), \end{aligned}$$

where the last module is projective over $A/A(1-e)A$, since $e\,\mathrm{rad}A$ is projective over A. This implies that $A/A(1-e)A$ is hereditary, and consequently, $\mathrm{gdim}(A/A(1-e)A) \le 1$ and $\mathrm{cd}(A/A(1-e)A) = 1$. Now the result follows immediately from Propositions 1.3(2) and 2.1. The proof of the corollary is completed.

We observe that the second inequality in Corollary 2.2(2) appears already in [6, Lemma 4] with the hypothesis that A be left serial and e be primitive. As another consequence of Proposition 2.1, the following result establishes immediately the Cartan determinant conjecture for quasi-stratified algebras.

2.3. COROLLARY. *If A is quasi-stratified, then* $\mathrm{cd}(A)$ *is positive.*

Proof. Assume that

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

is a quasi-stratification of A. Let e be a pseudo-primitive idempotent which generates the idempotent part of I_1. Note that $\mathrm{cd}(eAe) > 0$. If $r = 1$ then, by Lemma 1.10, e is primitive such that A is Morita equivalent to eAe. Thus $\mathrm{cd}(A) = \mathrm{cd}(eAe) > 0$. Assume now that $r > 1$. Then A/I_1 admits a quasi-stratification of length $r-1$, and by the induction hypothesis, $\mathrm{cd}(A/I_1) > 0$. By Proposition 2.1, we have $\mathrm{cd}(A) = \mathrm{cd}(eAe)\,\mathrm{cd}(A/I_1) > 0$. This completes the proof of the corollary.

2.4. LEMMA. *Let I be a quasi-stratifying ideal of A. Then A is of finite global dimension if and only if I is quasi-heredity and A/I is of finite global dimension.*

Proof. We may assume that I_A is projective. Let e be a pseudo-primitive idempotent which generates the idempotent part of I. By Corollary 1.4, A is of finite global dimension if and only if eAe and A/I are of finite global dimension. Being null or local, eAe is of finite global dimension if and only if $e\,\mathrm{rad}A\,e = 0$, that is, I is quasi-heredity. This completes the proof of the lemma.

We are now ready to get the main result of this section, which includes Wick's result on standardly stratified algebras; see [21, (1.7)].

2.5. THEOREM. *Let A be a quasi-stratified artin algebra. The following conditions are equivalent*:

(1) $\mathrm{cd}(A) = 1$.

(2) *A is of finite global dimension.*

(3) *A is ultimate-hereditary.*

Proof. Let

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

be a quasi-stratification of length r. We shall proceed by induction on r. Let e be a pseudo-primitive idempotent which generates the idempotent part of

I_1. If $r = 1$, then e is primitive. By Lemma 1.10, A is Morita equivalent to eAe and each of the three conditions stated in the theorem is equivalent to A being hereditary. Assume now that $r > 1$. Then A/I_1 admits a quasi-stratification of length $r - 1$. Moreover, it follows from Proposition 2.1 that $\mathrm{cd}(A) = \mathrm{cd}(eAe)\,\mathrm{cd}(A/I_1)$. Note that $\mathrm{cd}(eAe) = 1+c$, where c is the composition length of $e\,\mathrm{rad}A\,e$ as a right module over eAe.

Now $\mathrm{cd}(A) = 1$ if and only if $e\,\mathrm{rad}A\,e = 0$ and $\mathrm{cd}(A/I_1) = 1$. Since e is pseudo-primitive, $e\,\mathrm{rad}A\,e = 0$ if and only if $\mathrm{gdim}(eAe) < \infty$. Moreover, by the induction hypothesis, $\mathrm{cd}(A/I_1) = 1$ if and only if $\mathrm{gdim}(A/I_1) < \infty$. According to Corollary 1.4, we have the equivalence of (1) and (2).

If $\mathrm{gdim}(A) < \infty$, then by Lemma 2.4, I_1 is quasi-heredity and A/I_1 is finite global dimension. Applying the induction hypothesis, we infer that A/I_1 is ultimate-hereditary. Hence A is ultimate-hereditary by definition. This shows that (2) implies (3). Moreover, it follows from Proposition 1.11 that (3) implies (2). The proof of the theorem is now completed.

3. Self-extensions of simple modules

The objective of this section is to establish the no loop conjecture for quasi-stratified algebras, and the strong no loop conjecture for algebras which are quasi-stratified on one side. It is well known that if I is an idempotent projective ideal, then the extension groups of modules annihilated by I are preserved when one passes from A to A/I; see, for example, [11, Statement 4]. Unfortunately, this is no longer the case if I is not idempotent. Nevertheless, we have the following result.

3.1. Lemma. *Let I be a right projective ideal of A. If S is a simple right A/I-module, then*

$$\mathrm{Ext}^1_A(S,S) \cong \mathrm{Ext}^1_{A/I}(S,S).$$

Proof. Let S be a simple right A-module with $SI = 0$. First, we consider the case where I is nilpotent. Let

$$\eta: \quad 0 \longrightarrow S \longrightarrow E \longrightarrow S \longrightarrow 0$$

be in $\mathrm{Ext}^1_A(S,S)$. We shall show that $EI = 0$. Indeed, let $\{e_1, \ldots, e_n\}$ be a complete set of pairwise orthogonal primitive idempotents of A. We may assume that there exists some $1 \le r \le n$ such that $Se_i = S$ if and only if $1 \le i \le r$. Then $e_iA \cong e_1A$ if and only if $1 \le i \le r$. In particular, $Ee_j = 0$ for all $r < j \le n$. Let s be an integer with $1 \le s \le r$. Note that e_sI is a projective right A-module since I_A is projective. Moreover, $\ell\ell(e_sI_A) < \ell\ell(e_sA_A)$ since $I \subseteq \mathrm{rad}\,A$. Thus e_iA is not isomorphic to a direct summand of e_sI, for all $1 \le i \le r$. As a consequence, $e_sI \subseteq \sum_{r<j\le n} Ae_jA$, and hence $Ee_sI = 0$. This

shows that $EI = 0$, that is, $\eta \in \mathrm{Ext}^1_{A/I}(S,S)$. Hence the result is established in case I is nilpotent.

In general, let t be the idempotency of I. By Lemma 1.1, $J = I^t$ is an idempotent right projective ideal of A. Note that S is a simple A/J-module since $SJ \subseteq SI = 0$. Therefore, $\mathrm{Ext}^1_A(S,S) \cong \mathrm{Ext}^1_{A/J}(S,S)$. Moreover, I/J is a nilpotent right projective ideal of A/J such that $(A/J)/(I/J) \cong A/I$. It follows from our previous consideration that $\mathrm{Ext}^1_{A/J}(S,S) \cong \mathrm{Ext}^1_{A/I}(S,S)$. The proof of the lemma is completed.

The next lemma follows easily from [1, (2.4)]. However, we present a different argument here.

3.2. Lemma. *Let S be a simple right A-module of finite projective dimension, supported by a primitive idempotent e. If AeA is right projective, then* $\mathrm{Ext}^1_A(S,S) = 0$.

Proof. Let $\{e_1, \ldots, e_n\}$ with $e_1 = e$ be a basic set of primitive idempotents of A. For all $1 \le j \le n$, let c_j be the multiplicity of S as a composition factor of e_jA, which is equal to the composition length of e_jAe as a right module over eAe. Let

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to S \to 0$$

be a finite projective resolution of S. Write $P_i = (e_1A)^{r_{i1}} \oplus \cdots \oplus (e_nA)^{r_{in}}$ with $r_{ij} \ge 0$, for $i = 0, 1, \ldots, m$. It is well known that

$$1 = \sum_{i=0}^{m} (-1)^i (r_{i1}\, c_1 + \cdots + r_{in}\, c_n).$$

Assume that AeA is right projective. Then, for all $1 \le j \le n$, we have $e_jAeA \cong (eA)^{s_j}$ with $s_j \ge 0$. Hence $e_jAe \cong (eAe)^{s_j}$ as right eAe-modules, and consequently, $c_j = s_j\, c_1$ for all $1 \le j \le n$. This gives rise to

$$c_1 \sum_{i=1}^{m} (-1)^i (r_{i1}\, s_1 + \cdots + r_{in}\, s_n) = 1.$$

Thus $c_1 = 1$, and hence $e\, \mathrm{rad} A\, e = 0$. In particular, $\mathrm{Ext}^1_A(S,S) = 0$. The proof of the lemma is completed.

The following result can be considered as a weaker version of the strong no loop conjecture for quasi-stratified algebras.

3.3. Proposition. *Let A be a quasi-stratified algebra. If S is a simple* (*left or right*) *A-module with projective and injective dimensions finite, then* $\mathrm{Ext}^1_A(S,S) = 0$.

Proof. Let S be a simple right A-module with projective and injective dimensions finite. Then DS is a simple left A-module with projective and injective dimensions finite. Let

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

be a quasi-stratification of A, and let e be a pseudo-primitive idempotent which generates the idempotent part of I_1. We shall proceed by induction on r. If $r = 1$, then e is primitive. By Lemma 1.10, A is Morita equivalent to eAe. With the simple module having finite projective dimension, eAe is hereditary, and so is A. Thus $\mathrm{Ext}^1_A(S, S) = 0$.

Assume now that $r > 1$. Then A/I_1 admits a quasi-stratification of length $r - 1$. Let us consider the case where I_1 is left projective. If $SI_1 = 0$, then DS is a simple left module over A/I_1. It follows from Lemma 3.1 and the induction hypothesis that $\mathrm{Ext}^1_A(DS, DS) \cong \mathrm{Ext}^1_{A/I_1}(DS, DS) = 0$. If $SI_1 \neq 0$, then $S = SI_1 = SeA$, and hence DS is the simple left A-module supported by e. Since AeA is left projective by Lemma 1.1, $\mathrm{Ext}^1_A(DS, DS) = 0$ by Lemma 3.2. Therefore, $\mathrm{Ext}^1_A(S, S) = 0$ in both cases. The proof of the proposition is completed.

As an immediate consequence, we have the following result which excludes loops in the ordinary quiver of an ultimate-hereditary algebra.

3.4. Theorem. *Let A be a quasi-stratified algebra. If A is of finite global dimension, then* $\mathrm{Ext}^1_A(S, S) = 0$ *for all simple* (*left or right*) *A-modules S.*

Proof. If A is of finite global dimension, then every simple A-module has finite projective and injective dimensions. By Proposition 3.3, $\mathrm{Ext}^1_A(S, S) = 0$ for every simple A-module S. This completes the proof of the theorem.

Unfortunately, we need to put some restriction on a quasi-stratification in order to establish the strong no loop conjecture.

3.5. Definition. We say that A is *quasi-stratified on the right* (respectively, *left*) if A admits a quasi-stratification

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

such that I_{i+1}/I_i is a right (respectively, left) quasi-stratifying ideal of A/I_i, for all $0 \leq i < r$. Such a quasi-stratification is called a *right* (respectively, *left*) *quasi-stratification* of A.

It follows from the definition that a right standardly stratified algebra is quasi-stratified on the right.

3.6. Theorem. *Let A be an artin algebra which is quasi-stratified on the right. If S is a simple right A-module of finite projective dimension, then* $\mathrm{Ext}^1_A(S, S) = 0$.

Proof. Let S be a simple right A-module of finite projective dimension. Assume that

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A$$

is a right quasi-stratification of A. If $r = 1$ then, as we have seen in the proof of Proposition 3.3, A is hereditary. Thus $\mathrm{Ext}^1_A(S, S) = 0$. Suppose now that $r > 1$. If $SI_1 = 0$, then S is a simple right A/I_1-module, which is of finite projective dimension by Lemma 1.2(2). It follows from the induction hypothesis that $\mathrm{Ext}^1_{A/I_1}(S, S) = 0$. Thus $\mathrm{Ext}^1_A(S, S) = 0$, by Proposition 3.1. Otherwise, the idempotent part of I_1 is generated by a primitive idempotent e. Note that $S = SI_1 = SeA$. Since AeA is right projective by Lemma 1.1, it follows from Lemma 3.2 that $\mathrm{Ext}^1_A(S, S) = 0$. This completes the proof of the theorem.

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