CLUSTER CATEGORIES OF TYPE $\mathbb{A}_\infty^\infty$ AND TRIANGULATIONS OF THE INFINITE STRIP

SHIPING LIU AND CHARLES PAQUETTE

ABSTRACT. We call a 2-Calabi-Yau triangulated category a *cluster category* if its cluster-tilting subcategories form a cluster structure as defined in [4]. In this paper, we show that the canonical orbit category of the bounded derived category of finite dimensional representations of a quiver without infinite paths of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$ is a cluster category. Moreover, for a cluster category of type $\mathbb{A}_\infty^\infty$, we give a geometrical description of its cluster structure in terms of triangulations of an infinite strip with marked points in the plane.

INTRODUCTION

One of the most important developments of the representation theory of quivers is its interaction with cluster algebras, introduced by Fomin and Zelevinsky in connection with dual canonical bases and total positivity of semi-simple Lie groups; see [8, 9]. The two theories are linked together through cluster categories, constructed by Buan, Marsh, Reineke, Reiten and Todorov by taking the orbit category of the bounded derived category of finite dimensional representations of a finite acyclic quiver under the canonical auto-equivalence, that is the composite of the inverse of the Auslander-Reiten translation and the shift functor; see [5]. Such a cluster category is a categorification of the corresponding cluster algebra in such a way that cluster-tilting objects correspond to clusters and exchange of indecomposable summands of cluster-tilting objects correspond to mutations of cluster variables. These cluster categories are said to be of *finite rank* since every cluster-tilting object has only finitely many indecomposable summands. For cluster categories of type $\mathbb{A}_n$, Caldero, Chapoton and Schiffler gave a beautiful geometrical realization in terms of triangulations of an $(n+3)$-gon; see [7].

Replacing cluster-tilting objects by cluster-tilting subcategories, Buan, Iyama, Reiten and Scott introduced the notion of *cluster structure* in a 2-Calabi-Yau triangulated category; see, for example, (1.5). In this connection, we define a 2-Calabi-Yau triangulated category to be a *cluster category* if its cluster-tilting subcategories form a cluster structure. The first example of a cluster category of infinite rank was discovered by Holm and Jørgensen in [14], where they constructed a cluster

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category of type $\mathbb{A}_\infty$ as the finite derived category of dg-modules over the polynomial ring viewed as a dg-algebra and gave a geometrical realization of this cluster category in terms of triangulations of an infinity-gon.

The purpose of this paper is two-fold. Firstly we shall construct, following the canonical approach, cluster categories of types $\mathbb{A}_\infty$ and $\mathbb{A}_\infty^\infty$. Let Q be a locally finite quiver with no infinite path. The category $\operatorname{rep}(Q)$ of finite dimensional representations of Q is a hereditary abelian category such that $D^b(\operatorname{rep}(Q))$ has almost split triangles; see [3, (7.11)]. Thus, the canonical orbit category $\mathscr{C}(Q)$ of $D^b(\operatorname{rep}(Q))$ as mentioned above is a 2-Calabi-Yau triangulated category; see [19], and hence, it serves as a natural candidate for a cluster category of type Q. Indeed, the Auslander-Reiten components of $\mathscr{C}(Q)$ consists of a connecting component of shape $\mathbb{Z}Q^{\rm op}$, where $Q^{\rm op}$ denotes the opposite quiver of Q, and some possible regular components of shape $\mathbb{Z}\mathbb{A}_\infty$; see (2.5). Moreover, the projective representations in $\operatorname{rep}(Q)$ generate a cluster-tilting subcategory of $\mathscr{C}(Q)$; see (2.8). Therefore, in order to show that $\mathscr{C}(Q)$ is a cluster category, it suffices to verify that the quiver of every cluster-tilting subcategory of $\mathscr{C}(Q)$ has no oriented cycle of length one or two; see [4, (II.1.6)]. We conjecture that this is always the case. However, we shall prove it only in case Q is of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$. In this case, all the Auslander-Reiten components of $D^b(\operatorname{rep}(Q))$ are standard of shapes $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$; see [24, (2.2)]. In general, morphisms between objects in such components can be described in a pure combinatorial way; see (2.5). This enables us to show in this case that $\mathscr{C}(Q)$ is a cluster category; see (2.13), in which weakly cluster-tilting subcategories coincide with maximal rigid ones if Q is of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$; see (2.11).

Secondly, as an analogy to the above mentioned work by Caldero-Chapoton-Schiffler and by Holm-Jørgensen, we shall give a geometrical realization of a cluster category of type $\mathbb{A}_\infty^\infty$. For this purpose, we study in Section 3 triangulations of an infinite strip with marked points $\mathcal{B}_\infty$ in the plane. We introduce the notion of a *compact* triangulation; see (3.11) and give an easy criterion for a triangulation to be compact; see (3.18). In Section 4, we shall parameterize the indecomposable objects of $\mathscr{C}(Q)$ by the arcs in $\mathcal{B}_\infty$ in such a way that rigid pairs of indecomposable objects correspond to non-crossing pairs of arcs; see (4.3). In particular, weakly cluster-tilting subcategories of $\mathscr{C}(Q)$ correspond to triangulations of $\mathcal{B}_\infty$, and the functorial finiteness of a weakly cluster-tilting subcategory will be characterized by the compactness of the corresponding triangulation; see (4.7). This yields a geometric description of the cluster-tilting subcategories of $\mathscr{C}(Q)$. Finally, we would like to mention that triangulations of $\mathcal{B}_\infty$ have already been considered in [15, 16] as a geometrical model of a class of cluster categories constructed in a different approach.

We conclude with some new developments of cluster algebras of infinite rank. As a decategorification of Holm and Jørgensen's cluster category, Grabowski and Gratz constructed a cluster algebra of infinite rank as the coordinate ring of an infinite Grassmannian; see [11]. Moreover, to each simple Lie algebra, Hernandez and Leclerc associated some infinite quivers, from which they constructed a cluster algebra of infinite rank in order to study the representation theory of the corresponding untwisted quantum affine algebra; see [13]. We hope that our work would motivate further study on cluster categories or cluster algebras associated with infinite quivers.

1. Preliminaries

Throughout this paper, k stands for an algebraically closed field. All categories are k-linear with finite dimensional Hom-spaces over k. The standard duality for the category of finite dimensional k-spaces will be denoted by D. We refer to [1, Section 2] for the Auslander-Reiten theory of irreducible morphisms and almost split sequences in an abelian category, and to [12, Section 4] for that of irreducible morphisms and almost split triangles in a triangulated category.

Throughout this section, $\mathcal{A}$ stands for a Hom-finite Krull-Schmidt triangulated k-category having almost split triangles. That is, every indecomposable object of $\mathcal{A}$ is the starting term, as well as an ending term, of an almost split triangle. The Auslander-Reiten quiver $\varGamma_{\mathcal{A}}$ of $\mathcal{A}$ is a translation quiver, whose vertex set is chosen to be a complete set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{A}$ and whose translation is given by the Auslander-Reiten translation $\tau_{\mathcal{A}}$. If no confusion is possible, we shall write τ for $\tau_{\mathcal{A}}$. A path

$$M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{n-1} \longrightarrow M_n$$

in $\varGamma_{\mathcal{A}}$ is called *sectional* if there exists no i with $0 < i < n$ such that $\tau M_{i+1} = M_{i-1}$; and *almost sectional* if there exists at most one i with $0 < i < n$ such that $\tau M_{i+1} = M_{i-1}$. Let $\varGamma$ be a connected component of $\varGamma_{\mathcal{A}}$. One considers the *path category* $k\varGamma$ and the *mesh category* $k(\varGamma)$, where $k(\varGamma)$ is the quotient category of $k\varGamma$ modulo the ideal generated by the mesh elements in $k\varGamma$; see [26, (2.1)]. One says that $\varGamma$ is *standard* if $k(\varGamma)$ is equivalent to the full subcategory $\mathcal{A}(\varGamma)$ of $\mathcal{A}$ generated by the objects lying in $\varGamma$; see [26, (2.3)].

Given a quiver $\varDelta = (\varDelta_0, \varDelta_1)$ without oriented cycles, where $\varDelta_0$ is the vertex set and $\varDelta_1$ is the set of arrows, one constructs a stable translation quiver $\mathbb{Z}\varDelta$ in a canonical way; see [26, Page 47]. In the sequel, we shall denote by $\mathbb{N}\varDelta$ and by $\mathbb{N}^{-}\varDelta$ the full subquivers of $\mathbb{Z}\varDelta$ generated respectively by the vertices (n, x) and by the vertices $(-n, x)$, where $n \in \mathbb{N}$ and $x \in \varDelta_0$. Moreover, we shall say that $\varDelta$ is of type $\mathbb{A}$ if the underlying graph of $\varDelta$ is $\mathbb{A}_n$ with $n \geq 1$, or $\mathbb{A}_\infty$, or $\mathbb{A}_\infty^\infty$. In this case, $\mathbb{Z}\varDelta$ will be simply written as $\mathbb{Z}\mathbb{A}$.

Let $\varGamma$ be a connected component of $\varGamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}$. A *monomial mesh relation* in $\varGamma$ is a path $\tau X \to Y \to X$, where Y is the only immediate predecessor of X in $\varGamma$. Given $X \in \varGamma$, we define the *forward rectangle* $\mathscr{R}^X$ of X to be the full subquiver of $\varGamma$ generated by its successors Y such that no path $X \rightsquigarrow Y$ contains a monomial mesh relation. Dually, we define the *backward rectangle* $\mathscr{R}_X$ of X in $\varGamma$. If $\varGamma$ is of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$ then, by definition, $\mathscr{R}^X$ is generated by the successors of X and $\mathscr{R}_X$ is generated by the predecessors of X. The following result is well known for the $\mathbb{A}_n$-case.

1.1. Proposition. *Let $\varGamma$ be a standard component of $\varGamma_{\mathcal{A}}$ of shape $\mathbb{Z}\mathbb{A}$. If X, Y are objects in $\varGamma$, then $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$ if and only if $Y \in \mathscr{R}^X$ if and only if $X \in \mathscr{R}_Y$; and in this case, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is one-dimensional over k.*

Proof. We may assume with loss of generality that $\mathcal{A}(\varGamma) = k(\varGamma)$. For $u \in k\varGamma$, we write $\bar{u}$ for its image in $k(\varGamma)$. Let $X, Y \in \varGamma$. Clearly, $Y \in \mathscr{R}^X$ if and only if $X \in \mathscr{R}_Y$. If $p : X \rightsquigarrow Y$ and $q : X \rightsquigarrow Y$ are two parallel paths in $\varGamma$, then it is easy to see that $\bar{p} = \bar{q}$. Thus, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is at most one dimensional. It remains to prove the first equivalence stated in the proposition. Suppose that $Y \notin \mathscr{R}^X$. Then either

Y is not a successor of X in Γ, or else, Γ has a path $p : X \rightsquigarrow Y$ which contains a monomial mesh relation. In the first case, $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$. In the second case, $\bar{p} = 0$, and by the previously stated remark, $\bar{q} = 0$ for every path $q : X \rightsquigarrow Y$. As a consequence, $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$.

Suppose now that $Y \in \mathscr{R}^X$. Observe that all the paths from X to Y in Γ have the same length, written as $d(X, Y)$. We need to show that $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$, or equivalently, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is one dimensional. This is evident if $d(X, Y) = 0$. Assume that $d(X, Y) > 0$. Consider an almost split triangle

$$Z \longrightarrow U_1 \oplus U_2 \longrightarrow Y \longrightarrow Z[1]$$

in $\mathcal{A}$, where $U_1 \in \mathscr{R}^X$ and U_2 being zero or an object in Γ. By the induction hypothesis, $\mathrm{Hom}_{\mathcal{A}}(X, U_1)$ is one dimensional. Since $X \neq Y$, applying $\mathrm{Hom}_{\mathcal{A}}(X, -)$ to the almost split triangle yields an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(X, Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, U_1 \oplus U_2) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X, Y) \longrightarrow 0.$$

If $Z \notin \mathscr{R}^X$, then $\mathrm{Hom}_{\mathcal{A}}(X, Z) = 0$ and $\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, U_1 \oplus U_2) \neq 0$. Otherwise, by the definition of $\mathscr{R}^X$, we obtain $U_2 \neq 0$, and hence, $U_2 \in \mathscr{R}^X$. Since each of $\mathrm{Hom}_{\mathcal{A}}(X, Z)$, $\mathrm{Hom}_{\mathcal{A}}(X, U_1)$ and $\mathrm{Hom}_{\mathcal{A}}(X, U_2)$ is one-dimensional, we obtain $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$. The proof of the proposition is completed.

Let $\mathcal{T}$ be a full subcategory of $\mathcal{A}$. Given $X \in \mathcal{A}$, a morphism $f : X \to T$ with $T \in \mathcal{T}$ is called a *left $\mathcal{T}$-approximation* for X if every morphism $g : X \to M$ with $M \in \mathcal{T}$ factors through f. Dually, one defines a *right $\mathcal{T}$-approximation* for X. One says that $\mathcal{T}$ is *covariantly* (respectively, *contravariantly*) *finite* in $\mathcal{A}$ if every object in $\mathcal{A}$ admits a left (respectively, right) $\mathcal{T}$-approximation; and *functorially finite* in $\mathcal{A}$ if it is covariantly and contravariantly finite in $\mathcal{A}$. We say that $\mathcal{T}$ is *covariantly* (respectively, *contravariantly*) *bounded* in $\mathcal{A}$ provided that, for every $X \in \mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, M) = 0$ (respectively, $\mathrm{Hom}_{\mathcal{A}}(M, X) = 0$) for all but finitely many non-isomorphic indecomposable objects M of $\mathcal{T}$. The following statement follows easily from the Hom-finiteness of $\mathcal{A}$.

1.2. LEMMA. *A covariantly* (*respectively, contravariantly*) *bounded subcategory of $\mathcal{A}$ is covariantly* (*respectively, contravariantly*) *finite.*

Recall that $\mathcal{A}$ is called 2-*Calabi-Yau* if, for each pair of objects (X, Y), there exists an isomorphism $\mathrm{Hom}_{\mathcal{A}}(X, Y[1]) \cong D\mathrm{Hom}_{\mathcal{A}}(Y, X[1])$, which is natural in X and in Y. In this case, the Auslander-Reiten translation $\tau_{\mathcal{A}}$ coincides with the shift functor of $\mathcal{A}$; see [25, (I.2.3)]. We recall from [4] the following definition, where a *strictly additive* subcategory of $\mathcal{A}$ is a full subcategory closed under isomorphisms, finite direct sums, and taking summands.

1.3. DEFINITION. Let $\mathcal{A}$ be a 2-Calabi-Yau triangulated category with a strictly additive subcategory $\mathcal{T}$. One says that $\mathcal{T}$ is *weakly cluster-tilting* provided, for every $X \in \mathcal{A}$, that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, X[1]) = 0$ if and only if $X \in \mathcal{T}$; and *cluster-tilting* provided that $\mathcal{T}$ is weakly cluster-tilting and functorially finite in $\mathcal{A}$.

Let $\mathcal{T}$ be a strictly additive subcategory of $\mathcal{A}$. In particular, $\mathcal{T}$ is Krull-Schmidt. A morphism $f : X \to Y$ in $\mathcal{T}$ is called *right almost split* if it is not a retraction and every non-retraction morphism $g : M \to Y$ in $\mathcal{T}$ factors through f; *right minimal* if every factorization $f = fh$ implies that h is an automorphism; and a *sink morphism* in $\mathcal{T}$ if it is right minimal and right almost split in $\mathcal{T}$. Dually, we define a *left almost split, left minimal* or *source* morphism in $\mathcal{T}$. The *quiver* $Q_{\mathcal{T}}$ of $\mathcal{T}$ is defined so that its vertices form a complete set of representatives of the indecomposable objects of $\mathcal{T}$, and the number of arrows from a vertex X to a vertex Y is the k-dimension of $\mathrm{rad}_{\mathcal{T}}(X,Y)/\mathrm{rad}^2_{\mathcal{T}}(X,Y)$, where $\mathrm{rad}_{\mathcal{T}}(X,Y)$ denotes the k-space of morphisms in the Jacobson radical of $\mathcal{T}$. Moreover, given an indecomposable object M of $\mathcal{T}$, we shall denote by $\mathcal{T}_M$ the full additive subcategory of $\mathcal{T}$ generated by the indecomposable objects not isomorphic to M. Observe that $\mathcal{T}_M$ is also strictly additive in $\mathcal{A}$.

1.4. PROPOSITION. *Let $\mathcal{A}$ be a Hom-finite 2-Calabi-Yau triangulated k-category. If $\mathcal{T}$ is a cluster-tilting subcategory of $\mathcal{A}$, then it has source morphisms and sink morphisms; and consequently, its quiver $Q_{\mathcal{T}}$ is locally finite.*

Proof. Let $\mathcal{T}$ be a cluster-tilting subcategory of $\mathcal{A}$. Suppose that M is an indecomposable object of $\mathcal{T}$. Then $\mathcal{T}_M$ is functorially finite in $\mathcal{A}$; see [18, (4.1)]. Let $f : X \to M$ be a right $\mathcal{T}_M$-approximation for M. Then we can decompose f as $f = (g, 0) : X = Y \oplus Z \to M$, where $g : Y \to M$ is right minimal; see [21, (1.2)]. Thus, g is a minimal right $\mathcal{T}_M$-approximation for M.

If $\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(M)) = 0$, then g is right almost split, and hence, a sink morphism for M in $\mathcal{T}$. Otherwise, choose a k-basis $\{h_1, \ldots, h_m\}$ of $\mathrm{rad}(\mathrm{End}_{\mathcal{A}}(M))$ and set $h = (h_1, \cdots, h_m) : M^m \to M$. Then every radical endomorphism of M factors through h. As a consequence, $u = (g, h) : Y \oplus M^m \to M$ is right almost split in $\mathcal{T}$. Again, $u = (v, 0) : N \oplus L \to M$, where $v : N \to M$ is right minimal. Note that v is also right almost split, and hence, a sink morphism for M in $\mathcal{T}$. Dually, M admits a source morphism in $\mathcal{T}$. The proof of the proposition is completed.

We shall reformulate the notion of a cluster structure in a 2-Calabi-Yau triangulated category, which is originally introduced in [4, (II.1)].

1.5. DEFINITION. Let $\mathcal{A}$ be a 2-Calabi-Yau triangulated k-category. A non-empty collection $\mathfrak{C}$ of strictly additive subcategories of $\mathcal{A}$ is called a *cluster structure* if, for each subcategory $\mathcal{T} \in \mathfrak{C}$ and each indecomposable object $M \in \mathcal{T}$, the following conditions are verified.

(1) There exists a unique (up to isomorphism) indecomposable object M^* of $\mathcal{A}$, with $M^* \ncong M$, such that the additive subcategory $\mu_{_M}(\mathcal{T})$ of $\mathcal{A}$ generated by $\mathcal{T}_M$ and M^* belongs to $\mathfrak{C}$.
(2) There exist two exact triangles in $\mathcal{A}$ as follows:
$$M \xrightarrow{f} N \xrightarrow{g} M^* \longrightarrow M[1] \quad \text{and} \quad M^* \xrightarrow{u} L \xrightarrow{v} M \longrightarrow M^*[1],$$
where f, u are minimal left $\mathcal{T}_M$-approximations, and g, v are minimal right $\mathcal{T}_M$-approximations in $\mathcal{A}$.
(3) The quiver of $\mathcal{T}$ contains no oriented cycle of length one or two, from which the quiver of $\mu_{_M}(\mathcal{T})$ is obtained by the Fomin-Zelevinsky mutation at M as described in [9, (1.1)].

The following notion is the main objective of study of this paper.

1.6. DEFINITION. A 2-Calabi-Yau triangulated k-category is called a *cluster category* if its cluster-tilting subcategories form a cluster structure.

2. Cluster categories of types $\mathbb{A}_\infty$ and $\mathbb{A}_\infty^\infty$

As the main objective of this section, we shall show that the canonical orbit category of the bounded derived category of finite dimensional representations of a quiver without infinite paths of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$ is a cluster category.

We start with representations of quivers. Let Q be a connected locally finite quiver without infinite paths. By König's Lemma; see [20], the number of paths between every pair of vertices is finite. By definition, Q is strongly locally finite; see [3, Section 1]. Since Q has no infinite path, the category $\operatorname{rep}(Q)$ of finite dimensional k-linear representations of Q coincides with the category of finitely presented k-linear representations of Q; see [3, (1.5)]. Thus, $\operatorname{rep}(Q)$ is a hereditary abelian category having almost split sequences; see [3, (3.7)]. The vertex set of the Auslander-Reiten quiver $\Gamma_{\operatorname{rep}(Q)}$ of $\operatorname{rep}(Q)$ is chosen to contain the indecomposable projective representations P_x, the indecomposable injective representations I_x and the simple representations S_x, with $x \in Q_0$, as defined in [3, Section 1]. Its Auslander-Reiten translation is written as τ_Q. It is known that $\Gamma_{\operatorname{rep}(Q)}$ has a *preprojective* component $\mathcal{P}_Q$ which is standard of shape $\mathbb{N}Q^{\rm op}$ and contains all the P_x with $x \in Q_0$; and a *preinjective* component $\mathcal{I}_Q$ which is standard of shape $\mathbb{N}^-Q^{\rm op}$ and contains all the I_x with $x \in Q_0$. The other components of $\Gamma_{\operatorname{rep}(Q)}$ are called *regular*, which are of shape $\mathbb{Z}\mathbb{A}_\infty$; see [3, (4.16)] and [24, (2.2)]. Given two connected components Γ, Ω of $\Gamma_{\operatorname{rep}(Q)}$, we shall write $\operatorname{Hom}_{\operatorname{rep}(Q)}(\Gamma, \Omega) = 0$ if $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, Y) = 0$ for all $X \in \Gamma$ and $Y \in \Omega$; and say that Γ, Ω are *orthogonal* if $\operatorname{Hom}_{\operatorname{rep}(Q)}(\Gamma, \Omega) = 0$ and $\operatorname{Hom}_{\operatorname{rep}(Q)}(\Omega, \Gamma) = 0$.

2.1. Lemma. *Let Q be a connected locally finite quiver with no infinite path. Then* $\operatorname{Hom}_{\operatorname{rep}(Q)}(\mathcal{I}_Q, \mathcal{P}_Q) = 0$. *Moreover, if $\mathcal{R}$ is a regular component of $\Gamma_{\operatorname{rep}(Q)}$, then* $\operatorname{Hom}_{\operatorname{rep}(Q)}(\mathcal{I}_Q, \mathcal{R}) = 0$ *and* $\operatorname{Hom}_{\operatorname{rep}(Q)}(\mathcal{R}, \mathcal{P}_Q) = 0$.

Proof. Let $f : M \to N$ be a non-zero morphism with $M, N \in \Gamma_{\operatorname{rep}(Q)}$. Assume that M is preinjective, that is, $M = \tau_Q^r I_x$ for some $x \in Q_0$ and $r \in \mathbb{N}$. If N is not preinjective, then $N = \tau_Q^r L$ for some non-injective representation $L \in \Gamma_{\operatorname{rep}(Q)}$. Applying τ_Q^- yields a non-zero morphism $g : I_x \to L$; see [24, (2.1)], contrary to $\operatorname{rep}(Q)$ being hereditary. Dually, if N is preprojective, then so is M. The proof of the lemma is completed.

In case Q is of infinite Dynkin type, that is, the underlying graph of Q is $\mathbb{A}_\infty$, $\mathbb{A}_\infty^\infty$ or $\mathbb{D}_\infty$, the morphisms are better understood. Recall that the *support* $\operatorname{supp}(M)$ of a representation M is the set of vertices $x \in Q_0$ for which $M(x) \neq 0$.

2.2. Lemma. *Let Q be an infinite Dynkin quiver with no infinite path, and let X, Y be representations lying in $\Gamma_{\operatorname{rep}(Q)}$.*

(1) *If $X \neq Y$, then* $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, Y) = 0$ *or* $\operatorname{Hom}_{\operatorname{rep}(Q)}(Y, X) = 0$.

(2) *If Q is of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$, then* $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, Y)$ *is at most one-dimensional.*

Proof. Statement (1) follows from Lemma 2.1 and that every connected component of $\Gamma_{\operatorname{rep}(Q)}$ is standard without oriented cycles; see [3, (4.16)] and [24, (2.2)]. Assume that Q is of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$. Let Δ be a finite connected full subquiver of Q, containing the support of $X \oplus Y$. Then Δ is of type $\mathbb{A}_n$ for some n such that $\operatorname{Hom}_{\operatorname{rep}(Q)}(X, Y) \cong \operatorname{Hom}_{\operatorname{rep}(\Delta)}(X, Y)$, which is at most one-dimensional; see [10, (6.5)], and also, (1.1). The proof of the lemma is completed.

Let Γ be a connected component of $\Gamma_{\mathrm{rep}(Q)}$ of shape $\mathbb{Z}\mathbb{A}_\infty$, and let $X \in \Gamma$. One says that X is *quasi-simple* if it has only one immediate predecessor in Γ. In general, Γ has a unique sectional path $X = X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1$, where X_1 is quasi-simple. One defines the *quasi-length* $\ell(X)$ of X to be n.

Let Q be of type $\mathbb{A}_\infty^\infty$. We shall describe the quasi-simple representations in the regular component. To this end, we recall some terminology and notations. A *string* in Q is a finite reduced walk w, to which one associates a *string representation* $M(w)$; see [3, Section 5]. Let a_i, b_i, $i \in \mathbb{Z}$, be the source vertices and the sink vertices of Q, respectively, such that there exist paths $p_i : a_i \rightsquigarrow b_i$ and $q_i : a_i \rightsquigarrow b_{i-1}$, for $i \in \mathbb{Z}$. A vertex on a path is called a *middle point* if it is not an endpoint. Let Q_R denote the union of the p_i, $i \in \mathbb{Z}$, and the trivial paths ε_a, where a is a middle point of some q_j with $j \in \mathbb{Z}$. Dually, Q_L denotes the union of the q_i, $i \in \mathbb{Z}$, and the trivial paths ε_b, where b is a middle point of some p_j with $j \in \mathbb{Z}$. It is known that $\Gamma_{\mathrm{rep}(Q)}$ has exactly two regular components $\mathcal{R}_R$ and $\mathcal{R}_L$ such that the quasi-simple objects in $\mathcal{R}_R$ are the string representations $M(p)$ with $p \in Q_R$, and those in $\mathcal{R}_L$ are the string representations $M(q)$ with $q \in Q_L$; see [24, (2.2)] and [3, (5.16), (5.22)].

2.3. Lemma. *Let Q be a quiver of type $\mathbb{A}_\infty^\infty$ with no infinite path. Then the two regular components $\mathcal{R}_R$ and $\mathcal{R}_L$ of $\Gamma_{\mathrm{rep}(Q)}$ are orthogonal.*

Proof. Let $f : M \to N$ be a non-zero morphism with $M \in \mathcal{R}_R$ and $N \in \mathcal{R}_L$. We may assume that $m = \ell(M) + \ell(N)$ is minimal with respect to the existence of such a non-zero morphism. In view of the above description, the quasi-simple representations have pairwise disjoint supports. Thus, we may assume with no loss of generality that $\ell(N) > 1$. Then rep(Q) has a short exact sequence

$$0 \longrightarrow X \xrightarrow{\ u\ } N \xrightarrow{\ v\ } Y \longrightarrow 0,$$

where $X, Y \in \mathcal{R}_L$ with $\ell(X) = \ell(N) - 1$ and $\ell(Y) = 1$. By the minimality of m, we have $vf = 0$, and hence, $f = uw$ for some non-zero morphism $w : M \to X$, contrary to the minimality of m. The proof of the lemma is completed.

Let Γ be a connected component of $\Gamma_{\mathrm{rep}(Q)}$ of shape $\mathbb{Z}\mathbb{A}_\infty$, with a quasi-simple representation S. Observe that Γ has a unique infinite sectional path starting in S, called the *ray* starting in S and denoted by $(S\rightarrow)$; and a unique infinite sectional path ending in S, called the *co-ray* ending in S and denoted by $(\rightarrow S)$. Let $\mathcal{W}(S)$ be the full subquiver of Γ generated by the representations X for which there exist paths $M \rightsquigarrow X \rightsquigarrow N$, where M belongs to $(\rightarrow S)$ and N belongs to $(S\rightarrow)$. We call $\mathcal{W}(S)$ the *infinite wing* with *wing vertex* S; compare [26, (3.3)].

2.4. Proposition. *Let Q be a quiver of type $\mathbb{A}_\infty^\infty$ with no infinite path, and let $X \in \mathcal{P}_Q$. If $\mathcal{R}$ is a regular component of $\Gamma_{\mathrm{rep}(Q)}$, then it has a unique quasi-simple representation S such that, for every $Y \in \mathcal{R}$, $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, Y) \neq 0$ if and only if $Y \in \mathcal{W}(S)$; and every morphism $f : X \to Y$ factors through a representation belonging to the co-ray $(\rightarrow S)$.*

Proof. We keep the notation introduced above and assume that $\mathcal{R} = \mathcal{R}_R$. Applying τ_Q if necessary, we may assume that $X = P_x$ for some $x \in Q_0$; see [24, (2.1)]. Since $P_x \notin \mathcal{R}$, applying $\mathrm{Hom}_{\mathrm{rep}(Q)}(P_x, -)$ yields an additive function d on $\mathcal{R}$; see, for definition, [26, (A.1)], defined by $d(Y) = \dim_k \mathrm{Hom}_{\mathrm{rep}(Q)}(P_x, Y)$ for $Y \in \mathcal{R}$. Set $S = M(p)$, where p is the unique path in Q_R in which x appears. Then $d(S) = 1$

and $d(\tau^i S) = 0$ for all $i \neq 0$. Using the additivity of d, we see first that $d(Y) = 0$ for all $Y \notin \mathcal{W}(S)$, and then $d(Y) = 1$ for all $Y \in \mathcal{W}(S)$.

Suppose now that $f : P_x \to Y$ is non-zero morphism with $Y \in \mathcal{W}(S)$. There exists a unique sectional path $p : Z \rightsquigarrow Y$ in $\mathcal{R}$ with Z belonging to $(\to S)$. Observe that there exists a monomorphism $g : Z \to Y$ in $\mathrm{rep}(Q)$. Since P_x is projective, we obtain a monomorphism $\mathrm{Hom}(P_x, g) : \mathrm{Hom}(P_x, Z) \to \mathrm{Hom}(P_x, Y)$, which is an isomorphism since $d(Y) = d(Z) = 1$. Hence, f factors through g. The proof of the proposition is completed.

Remark. The dual statement holds for preinjective representations.

Next, we consider the bounded derived category $D^b(\mathrm{rep}(Q))$ of $\mathrm{rep}(Q)$. We regard $\mathrm{rep}(Q)$ as a full subcategory of $D^b(\mathrm{rep}(Q))$ in a canonical way. It is known that $D^b(\mathrm{rep}(Q))$ is a Hom-finite Krull-Schmidt triangulated k-category having almost split triangles; see [3, (7.11)]. The vertices of its Auslander-Reiten quiver $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$ are chosen to be the shifts of the vertices of $\mathit{\Gamma}_{\mathrm{rep}(Q)}$. The Auslander-Reiten translation $\tau_{_D}$ is such that $\tau_{_D} X = \tau_{_Q} X$ for $X \in \mathit{\Gamma}_{\mathrm{rep}(Q)}$ non-projective, and $\tau_{_D} P_x = I_x[-1]$ for $x \in Q_0$. Thus, $\tau_{_D}$ induces an auto-equivalence of $D^b(\mathrm{rep}(Q))$. A regular component of $\mathit{\Gamma}_{\mathrm{rep}(Q)}$ is a connected component of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$, while $\mathcal{P}_Q$ and $\mathcal{I}_Q[-1]$ are glued together to form the *connecting* component $\mathcal{C}_Q$ of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$, which is of shape $\mathbb{Z}Q^{\mathrm{op}}$. The connected components of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$ are the shifts of $\mathcal{C}_Q$ and those of the regular components of $\mathit{\Gamma}_{\mathrm{rep}(Q)}$; see [3, (7.9),(7.10)].

Finally, we consider the canonical auto-equivalence $F = \tau_{_D}^{-1} \circ [1]$ of $D^b(\mathrm{rep}(Q))$. By a well known result of Keller; see [19, Section 9], the canonical orbit category

$$\mathscr{C}(Q) = D^b(\mathrm{rep}(Q))/F$$

is a Hom-finite Krull-Schmidt 2-Calabi-Yau triangulated k-category such that the canonical projection $\pi : D^b(\mathrm{rep}(Q)) \to \mathscr{C}(Q)$ is triangle-exact. We shall denote by $\tau_{_\mathscr{C}}$ the Auslander-Reiten translation of $\mathscr{C}(Q)$. The connected components of the Auslander-Reiten quiver $\mathit{\Gamma}_{\mathscr{C}(Q)}$ of $\mathscr{C}(Q)$ are described in the following result.

2.5. Theorem. *Let Q be an infinite connected quiver, which is locally finite and contains no infinite path.*

(1) *The canonical projection $\pi : D^b(\mathrm{rep}(Q)) \to \mathscr{C}(Q)$ sends Auslander-Reiten triangles to Auslander-Reiten triangles.*
(2) *If $\mathit{\Gamma}$ is a connected component of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$, then $\pi(\mathit{\Gamma})$ is a connected component of $\mathit{\Gamma}_{\mathscr{C}(Q)}$ such that $\pi(\mathit{\Gamma}) \cong \mathit{\Gamma}$ as translation quivers.*
(3) *The connected components of $\mathit{\Gamma}_{\mathscr{C}(Q)}$ are the components $\pi(\mathit{\Gamma})$, where $\mathit{\Gamma}$ is either the connecting component of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$ or a regular component of $\mathit{\Gamma}_{\mathrm{rep}(Q)}$.*

Proof. Observe that a rigorous definition of an orbit category of $D^b(\mathrm{rep}(Q))$ requires an automorphism of $D^b(\mathrm{rep}(Q))$. In order to overcome this problem, we shall take a skeleton $\mathscr{D}$ of $D^b(\mathrm{rep}(Q))$, containing the vertices of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$. Then $\mathscr{D}$ is a Hom-finite Krull-Schmidt triangulated k-category such that the inclusion functor $\mathscr{D} \to D^b(\mathrm{rep}(Q))$ is a triangle-equivalence and $\mathit{\Gamma}_{\mathscr{D}} = \mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$. Observe that the translation $\tau_{_D}$ of $\mathit{\Gamma}_{D^b(\mathrm{rep}(Q))}$ induces an automorphism of $\mathscr{D}$, which is denoted again by $\tau_{_D}$. Setting $F = \tau_{_D}^{-1} \circ [1]$, we obtain a group $G = \{F^n \mid n \in \mathbb{Z}\}$ of automorphisms of $\mathscr{D}$. Observe that the action of G on $\mathscr{D}$ is free and locally bounded, that is, no indecomposable object is fixed by any non-identity element of G; and

$\mathrm{Hom}_{\mathscr{D}}(X, F^iY) = 0$ for all but finitely many integers i; see [2, (2.1)]. Now, the image $\mathscr{C}$ of $\mathscr{D}$ under the canonical projection $\pi : D^b(\mathrm{rep}(Q)) \to \mathscr{C}(Q)$ is a dense full triangulated subcategory of $\mathscr{C}(Q)$. In particular, $\mathscr{C}$ is Hom-finite and Krull-Schmidt. Restricting $\pi : D^b(\mathrm{rep}(Q)) \to \mathscr{C}(Q)$, we obtain a triangle functor $\mathscr{D} \to \mathscr{C}$, which is denoted again by π. For $X \in \mathscr{D}$ and $n \in \mathbb{Z}$, we define

$$\delta_{n,X} = (\delta_{n,i})_{i\in\mathbb{Z}} \in \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}}(F^nX, F^iX) = \mathrm{Hom}_{\mathscr{C}}(F^nX, X),$$

where $\delta_{n,i} = \mathbf{1}_{F^nX}$ if $i = n$; otherwise, $\delta_{n,i} = 0$. It is easy to see that $\delta_{n,X}$ is an isomorphism, which is natural in X, such that $\delta_{n,X} \circ \delta_{m,F^nX} = \delta_{n+m,X}$, for integers m, n. This yields functorial isomorphisms $\delta_n : \pi \circ F^n \to \pi$, $n \in \mathbb{Z}$, such that $\delta = (\delta_n)_{n\in\mathbb{Z}}$ is a G-stabilizer for π; see [2, (2.3)]. It is not hard to verify that

$$\pi_{X,Y} : \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}}(X, F^iY) \to \mathrm{Hom}_{\mathscr{C}}(X, Y) : (f_i)_{i\in\mathbb{Z}} \mapsto \textstyle\sum_{i\in\mathbb{Z}} \delta_{i,Y} \circ \pi(f_i)$$

is the identity map. Hence, π is a G-precovering; see [2, (2.5)]. Since $\mathscr{D}$ is evidently Hom-finite and Krull-Schmidt, $\pi : \mathscr{D} \to \mathscr{C}$ a Galois G-covering; see [2, (2.8), (2.9)]. By Proposition 3.5 in [2], the exact functor $\pi : \mathscr{D} \to \mathscr{C}$ sends Auslander-Reiten triangles to Auslander-Reiten triangles, and hence, Statement (1) holds.

Observe that $\Gamma_{\mathscr{C}(Q)} = \Gamma_{\mathscr{C}}$. Let Γ be a connected component of $\Gamma_{\mathscr{D}}$. By Theorem 4.7 stated in [2], $\pi(\Gamma)$ is a connected component of $\Gamma_{\mathscr{C}}$ such that π restricts to Galois G_Γ-covering $\pi_{_\Gamma} : \Gamma \to \pi(\Gamma)$, where $G_\Gamma = \{F^n \mid F^n(\Gamma) = \Gamma\}$. Since Q is infinite, $F^n(\Gamma) \neq \Gamma$ for every $n \neq 0$. That is, G_Γ is trivial, and hence, $\pi_{_\Gamma}$ is an isomorphism of translation quivers; see [2, (4.6)]. This establishes Statement (2).

Since π is dense, $\Gamma_{\mathscr{C}}$ consists of the connected components $\pi(\Theta)$ with Θ ranging over the connected components of $\Gamma_{\mathscr{D}}$. If Θ is such a component, then $\Theta = F^n(\Gamma)$, where $n \in \mathbb{Z}$ and Γ is the connecting component of $\Gamma_{\mathscr{D}}$ or a connected component of $\Gamma_{\mathrm{rep}(Q)}$. This yields $\pi(\Theta) = \pi(\Gamma)$. The proof of the theorem is completed.

REMARK. (1) If $X \in \Gamma_{\mathrm{rep}(Q)}$ is non-projective, then $\tau_{_\mathscr{C}} X = \tau_{_D} X = \tau_{_Q} X$.

(2) By abuse of language and notation, we shall identify the connecting component $\mathcal{C}_Q$ of $\Gamma_{D^b(\mathrm{rep}(Q))}$ with $\pi(\mathcal{C}_Q)$ and call it the *connecting component* of $\Gamma_{\mathscr{C}(Q)}$, and identify a regular component $\mathcal{R}$ of $\Gamma_{\mathrm{rep}(Q)}$ with $\pi(\mathcal{R})$ and call it a *regular component* of $\Gamma_{\mathscr{C}(Q)}$.

(3) The set $\mathscr{F}(Q)$ of objects of $D^b(\mathrm{rep}(Q))$ lying in $\mathcal{C}_Q$ or a regular component of $\Gamma_{\mathrm{rep}(Q)}$ form a *fundamental domain* of $\mathscr{C}(Q)$. That is, every indecomposable object of $\mathscr{C}(Q)$ is isomorphic to a unique object in $\mathscr{F}(Q)$. Observe that every object in $\mathscr{F}(Q)$ lies in the $\tau_{_\mathscr{C}}$-orbit of a preprojective or regular representation in $\Gamma_{\mathrm{rep}(Q)}$.

We shall need the following description of the morphisms between objects in the fundamental domain of $\mathscr{C}(Q)$.

2.6. LEMMA. *Let Q be a locally finite quiver with no infinite path, and let X, Y be representations lying in $\Gamma_{\mathrm{rep}(Q)}$.*

(1) $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y) \cong \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X, Y) \oplus D\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(Y, \tau_{_D}^2 X)$.

(2) *If $X \notin \mathcal{I}_Q$ and $Y \in \mathcal{I}_Q$, then* $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y[-1]) \cong \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X, \tau_{_D}^- Y)$.

(3) *If $X \in \mathcal{I}_Q$ and $Y \notin \mathcal{I}_Q$, then* $\mathrm{Hom}_{\mathscr{C}(Q)}(X[-1], Y) \cong D\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(Y, \tau_{_D} X)$.

Proof. Since $D^b(\mathrm{rep}(Q))$ has almost split triangles, there exists a Serre duality

$$\mathrm{Hom}_{D^b(\mathrm{rep}(Q)}(M, N[1]) \cong D\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(N, \tau_{_D} M)$$

for all $M, N \in D^b(\mathrm{rep}(Q))$; see [25, (I.2.4)]. We shall prove only Statement (1), since the other two statements can be shown in a similar fashion. Since $\mathrm{rep}(Q)$ is hereditary, we deduce from the definition of F that

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) = \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,Y) \oplus \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,FY).$$

Since $FY = (\tau_D^- Y)[1]$, we deduce from the Serre duality that

$$\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,FY) \cong D\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(\tau_D^- Y, \tau_D X) \cong D\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(Y, \tau_D^2 X).$$

The proof of the lemma is completed.

The following consequence is useful for our future investigation.

2.7. Corollary. *Let Q be a locally finite quiver with no infinite path, and let $X \in \mathcal{P}_Q$ and $Y \in \Gamma_{\mathrm{rep}(Q)}$. If $X = P_x$ for some $x \in Q_0$ or $Y \notin \mathcal{P}_Q$, then*

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) \cong \mathrm{Hom}_{\mathrm{rep}(Q)}(X,Y).$$

Proof. We claim that $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(Y, \tau_D^2 X) = 0$. Indeed, since $\tau_D^2 P_x = \tau_Q I_x[-1]$, this is evident in case $X = P_x$. Assume that $Y \notin \mathcal{P}_Q$. Since $X \in \mathcal{P}_Q$, either $\tau_D^2 X \in \mathcal{I}_Q[-1]$ or $\tau_D^2 X \in \mathcal{P}_Q$. In the first case, the claim holds. In the second case, $\mathrm{Hom}_{\mathrm{rep}(Q)}(Y, \tau_D^2 X) = 0$ by Lemma 2.1. This establishes the claim. Now, the result follows from Lemma 2.6(1). The proof of the corollary is completed.

The following result shows the existence of cluster-tilting subcategories in $\mathscr{C}(Q)$.

2.8. Proposition. *Let Q be a locally finite quiver with no infinite path. The strictly additive subcategory $\mathscr{P}$ of $\mathscr{C}(Q)$ generated by the representations P_x with $x \in Q_0$ is cluster-tilting.*

Proof. Since $\mathscr{C}(Q)$ is 2-Calabi-Yau, the Auslander-Reiten translation $\tau_{\mathscr{C}}$ for $\mathscr{C}(Q)$ coincides with its shift functor. Given $x, y \in Q_0$, we have $\tau_{\mathscr{C}} P_y = I_y[-1]$ and $\tau_D^- I_y = P_y[1]$. In view of Lemma 2.6(2), we obtain

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}(Q)}(P_x, P_y[1]) &= \mathrm{Hom}_{\mathscr{C}(Q)}(P_x, I_y[-1]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(P_x, \tau_D^- I_y) \\ &= \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(P_x, P_y[1]), \\ &= 0. \end{aligned}$$

Let $X \in \mathscr{C}(Q)\backslash\mathscr{P}$. We may assume that $X \in \mathscr{F}(Q)$, the fundamental domain of $\mathscr{C}(Q)$. If $X \in \Gamma_{\mathrm{rep}(Q)}$, then $X[1] = \tau_{\mathscr{C}} X = \tau_Q X \in \Gamma_{\mathrm{rep}(Q)}$. Choosing $x \in \mathrm{supp}(\tau_Q X)$, by Lemma 2.6(1), we obtain $\mathrm{Hom}_{\mathscr{C}(Q)}(P_x, X[1]) \neq 0$. Otherwise, $X = Y[-1]$, for some $Y \in \mathcal{I}_Q$. Choosing $y \in \mathrm{supp}(Y)$, in view of Lemma 2.6(1), $\mathrm{Hom}_{\mathscr{C}(Q)}(P_y, X[1]) \neq 0$. Thus, $\mathscr{P}$ is weakly cluster-tilting.

Let $Z \in \mathscr{F}(Q)$. We claim that $\mathrm{Hom}_{\mathscr{C}(Q)}(Z, -)$ and $\mathrm{Hom}_{\mathscr{C}(Q)}(-, Z)$ vanish on all but finitely many indecomposable objects of $\mathscr{P}$. Suppose first that $Z \in \Gamma_{\mathrm{rep}(Q)}$. Then $\tau_D^2 Z = \tau_Q^2 Z$ if the latter is defined, and otherwise, $\tau_D^2 Z \in \mathcal{I}_Q[-1]$. Let $x \in Q_0$ be such that $\mathrm{supp}(P_x)$ intersects neither $\mathrm{supp}(Z)$ nor $\mathrm{supp}(\tau_Q^2 Z)$. By Corollary 2.7, $\mathrm{Hom}_{\mathscr{C}(Q)}(P_x, Z) = 0$, and by Lemma 2.6(1), $\mathrm{Hom}_{\mathscr{C}(Q)}(Z, P_x) = 0$. Similarly, we can establish the claim in case $Z \in \mathcal{I}_Q[-1]$. This shows that $\mathscr{P}$ is covariantly and contravariantly bounded in $\mathscr{C}(Q)$. By Lemma 1.2, $\mathscr{P}$ is functorially finite in $\mathscr{C}(Q)$. The proof of the proposition is completed.

For the rest of this section, we shall concentrate on the infinite Dynkin case.

2.9. PROPOSITION. *Let Q be an infinite Dynkin quiver with no infinite path. The connected components of $\Gamma_{\mathscr{C}(Q)}$ consist of the connecting component of shape $\mathbb{Z}Q^{\rm op}$ and r regular components of shape $\mathbb{Z}\mathbb{A}_\infty$, where*

(1) *$r = 0$ if Q is of type $\mathbb{A}_\infty$;*
(2) *$r = 1$ if Q is of type $\mathbb{D}_\infty$;*
(3) *$r = 2$ if Q is of type $\mathbb{A}_\infty^\infty$; and in this case, the two regular components are orthogonal.*

Proof. It is known that $\Gamma_{{\rm rep}(Q)}$ has $0, 1, 2$ regular components in case Q is of type $\mathbb{A}_\infty$, $\mathbb{D}_\infty$, $\mathbb{A}_\infty^\infty$, respectively; see [3, (5.16),(5.17),(5.22)]. All statements of the proposition, except for the second part of Statement (3), follow from Theorem 2.5. Suppose that Q is of type $\mathbb{A}_\infty^\infty$. Let $\mathcal{R}, \mathcal{S}$ be the two distinct regular components of $\Gamma_{{\rm rep}(Q)}$ which, by Proposition 2.3, are orthogonal in ${\rm rep}(Q)$. Let $X \in \mathcal{R}$ and $Y \in \mathcal{S}$. Since $\tau_{_D}^2 X = \tau_{_Q}^2 X$, we deduce from Lemma 2.6(1) that ${\rm Hom}_{\mathscr{C}(Q)}(X, Y) = 0$. The proof of the proposition is completed.

An object X of $\mathscr{C}(Q)$ is called a *brick* if ${\rm End}_{\mathscr{C}(Q)}(X)$ is one-dimensional over k; and *rigid* if ${\rm Hom}_{\mathscr{C}(Q)}(X, X[1]) = 0$.

2.10. COROLLARY. *Let Q be an infinite Dynkin quiver with no infinite path. Then every indecomposable object of $\mathscr{C}(Q)$ is a rigid brick.*

Proof. Let X be an indecomposable object of $\mathscr{C}(Q)$. Since $\tau_{_\mathscr{C}}$ is an auto-equivalence of $\mathscr{C}(Q)$, we may assume that $X, \tau_{_\mathscr{C}} X \in \Gamma_{{\rm rep}(Q)}$. Let Γ be the connected component of $\Gamma_{D^b({\rm rep}(Q))}$ containing X. Since Γ is standard with no oriented cycle; see [24, (2.3)], ${\rm Hom}_{D^b({\rm rep}(Q))}(X, \tau_{_D} X) = 0$, ${\rm Hom}_{D^b({\rm rep}(Q))}(X, \tau_{_D}^2 X) = 0$, and ${\rm End}_{D^b({\rm rep}(Q))}(X)$ is one-dimensional. Thus, ${\rm End}_{\mathscr{C}(Q)}(X)$ is one-dimensional by Lemma 2.6(1). Moreover, we have

$$\begin{array}{rcl} {\rm Hom}_{\mathscr{C}(Q)}(X, X[1]) & \cong & {\rm Hom}_{\mathscr{C}(Q)}(X, \tau_{_D} X) \\ & \cong & {\rm Hom}_{D^b({\rm rep}(Q))}(X, \tau_{_D} X) \oplus D{\rm Hom}_{D^b({\rm rep}(Q))}(\tau_{_D} X, \tau_{_D}^2 X) \\ & = & 0. \end{array}$$

The proof of the corollary is completed.

More generally, a strictly additive subcategory $\mathcal{T}$ of $\mathscr{C}(Q)$ is called *rigid* if ${\rm Hom}_{\mathscr{C}(Q)}(X, Y[1]) = 0$, for $X, Y \in \mathcal{T}$; and *maximal rigid* if it is rigid and maximal with respect to the rigidity property. A weakly cluster-tilting subcategory of $\mathscr{C}(Q)$ is clearly maximal rigid, and the converse is not true in general; see [6, (1.3)].

2.11. LEMMA. *Let Q be an infinite Dynkin quiver with no infinite path. If $\mathcal{T}$ is a strictly additive subcategory of $\mathscr{C}(Q)$, then it is weakly cluster-tilting if and only if it is maximal rigid in $\mathscr{C}(Q)$.*

Proof. Let $\mathcal{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$, which is maximal rigid. Let $M \in \mathscr{C}(Q)$ be indecomposable such that ${\rm Hom}_{\mathscr{C}(Q)}(\mathcal{T}, M[1]) = 0$. Since $\mathscr{C}(Q)$ is 2-Calabi-Yau, ${\rm Hom}_{\mathscr{C}(Q)}(M, \mathcal{T}[1]) = 0$. By Corollary 2.10, M is rigid in $\mathscr{C}(Q)$. Hence, the strictly additive subcategory of $\mathscr{C}(Q)$ generated by M and $\mathcal{T}$ is rigid. Since $\mathcal{T}$ is maximal rigid, $M \in \mathcal{T}$. The proof of the lemma is completed.

The following result is essential for our investigation.

2.12. PROPOSITION. *Let Q be a quiver with no infinite path of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$. If $X, Y \in \mathscr{C}(Q)$ are indecomposable, then $\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y)$ is at most one dimensional.*

Proof. Let $X, Y \in \mathscr{F}(Q)$. Since $\Gamma_{\mathscr{C}(Q)}$ is stable and $\tau_{\mathscr{C}}$ is an auto-equivalence of $\mathscr{C}$, we may assume that $X, Y \in \Gamma_{\mathrm{rep}(Q)}$. In particular, X, Y are preprojective or regular representations. If $X, Y \notin \mathcal{P}_Q$, then the result follows from Proposition 2.9. If $X \in \mathcal{P}_Q$ and $Y \notin \mathcal{P}_Q$, then the result follows from Corollary 2.7 and Lemma 2.2. If $X \notin \mathcal{P}_Q$ and $Y \in \mathcal{P}_Q$, then the result follows from Lemmas 2.1 and 2.6(1). Finally, assume that $X, Y \in \mathcal{P}_Q$. In particular, X, Y lie in the connecting component $\mathcal{C}_Q$ of $\Gamma_{D^b(\mathrm{rep}(Q))}$, which is standard without oriented cycles; see [3, (7.9)] and [24, (2.3)]. In particular, $\mathcal{C}_Q$ contains no path $X \rightsquigarrow Y$ or no path $Y \rightsquigarrow \tau_D^2 X$. Thus, $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,Y) = 0$ or $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(Y, \tau_Q^2 X) = 0$. Now, the result follows from Lemmas 2.2 and 2.6(1). The proof of the proposition is completed.

We are ready to present the main result of this section. We shall say that a pair (X, Y) of indecomposable objects of $\mathscr{C}(Q)$ is *rigid* if $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y[1]) = 0$.

2.13. THEOREM. *Let Q be a quiver of type $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$ with no infinite path. Then $\mathscr{C}(Q)$ is a cluster category.*

Proof. By Theorem II.1.6 in [4] and Lemmas 2.8 and 2.11, it suffices to show that the quiver of every cluster-tilting subcategory of $\mathscr{C}(Q)$ has no oriented cycle of length two. If this is not the case, then there exists a rigid pair (X, Y) of distinct indecomposable objects of $\mathscr{F}(Q)$ such that $\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) \neq 0$ and $\mathrm{Hom}_{\mathscr{C}(Q)}(Y,X) \neq 0$. Since $\tau_{\mathscr{C}}$ is an auto-equivalence of $\mathscr{C}(Q)$, we may assume that $\tau_{\mathscr{C}}^2 X, \tau_{\mathscr{C}}^2 Y \in \Gamma_{\mathrm{rep}(Q)}$. Then it follows from Lemma 2.6(1) that

$$\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) \cong \mathrm{Hom}_{\mathrm{rep}(Q)}(X,Y) \oplus D\mathrm{Hom}_{\mathrm{rep}(Q)}(Y, \tau_Q^2 X) \qquad (*)$$

Suppose that $\mathrm{Hom}_{\mathrm{rep}(Q)}(X,Y) \neq 0$. By Lemma 2.2(1), $\mathrm{Hom}_{\mathrm{rep}(Q)}(Y,X) = 0$. Then, $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, \tau_Q^2 Y) \neq 0$. Since $\mathrm{Hom}_{\mathscr{C}(Q)}(X, \tau_Q Y) = \mathrm{Hom}_{\mathscr{C}(Q)}(X, Y[1]) = 0$, we obtain $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, \tau_Q Y) = 0$.

Let Γ be the connected component of $\Gamma_{D^b(\mathrm{rep}(Q))}$ containing X. Then, Γ is standard of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty^\infty$; see [24, (2.3)] and [3, (7.9)]. If $Y \in \Gamma$, by Proposition 1.1, both $\tau_D^2 Y$ and Y lie in the forward rectangle $\mathscr{R}^X$ of X. Being convex, $\mathscr{R}^X$ also contains $\tau_D Y$. Applying again Proposition 1.1, $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X, \tau_D Y) \neq 0$, a contradiction. Therefore, Y lies in a connected component Ω of $\Gamma_{D^b(\mathrm{rep}(Q))}$ different from Γ. Then, Q is of type $\mathbb{A}_\infty^\infty$ by Proposition 2.9. Since X, Y are preprojective or regular representations, by Lemma 2.1, $X \in \mathcal{P}_Q$ and Y is regular. This implies that Ω is a regular component of $\Gamma_{\mathrm{rep}(Q)}$. By Proposition 2.4, Ω has an infinite wing $\mathcal{W}(S)$ such that, for each $Z \in \Omega$, we have $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, Z) \neq 0$ if and only if $Z \in \mathcal{W}(S)$. In particular, $Y, \tau_Q^2 Y \in \mathcal{W}(S)$, and consequently, $\tau_Q Y \in \mathcal{W}(S)$. That is, $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, \tau_Q Y) \neq 0$, a contradiction. Thus, $\mathrm{Hom}_{\mathrm{rep}(Q)}(X,Y) = 0$.

Similarly, we can show that $\mathrm{Hom}_{\mathrm{rep}(Q)}(Y,X) = 0$. In view of the isomorphism $(*)$, we obtain $\mathrm{Hom}_{\mathrm{rep}(Q)}(Y, \tau_Q^2 X) \neq 0$ and $\mathrm{Hom}_{\mathrm{rep}(Q)}(X, \tau_Q^2 Y) \neq 0$. Since every connected component of $\Gamma_{\mathrm{rep}(Q)}$ is standard without oriented cycles, X and Y lie in two different connected components of $\Gamma_{\mathrm{rep}(Q)}$. Since X, Y are preprojective or regular, by Lemma 2.1, both X and Y are regular. Then Q is of type $\mathbb{A}_\infty^\infty$ by Proposition 2.9 and this contradicts Proposition 2.3. The proof of the theorem is completed.

3. Triangulations of an infinite strip with marked points

The objective of this section is to study triangulations of an infinite strip with marked points in the plane, which will serve as a geometric model for our cluster categories of type $\mathbb{A}_\infty^\infty$; compare [15, 16].

For the rest of this paper, we denote by $\mathcal{B}_\infty$ the infinite strip in the plane of the points (x, y) with $0 \le y \le 1$. The points $\mathfrak{l}_i = (i, 1)$, $i \in \mathbb{Z}$, are called the *upper marked points*; and $\mathfrak{r}_i = (-i, 0)$, $i \in \mathbb{Z}$, the *lower marked points*. An upper or lower marked point will be simply called a *marked point*. By a *simple curve* in $\mathcal{B}_\infty$ we mean a curve which does not cross itself and joins two (maybe identical) marked points called *endpoints*. A simple curve is called *internal* if it intersects the boundary of $\mathcal{B}_\infty$ only at the endpoints. Two distinct simple curves in $\mathcal{B}_\infty$ are said to *cross* if they have a common point which is not an endpoint of any of the curves.

Let $\mathfrak{p}, \mathfrak{q}$ be distinct marked points in $\mathcal{B}_\infty$. There exists a unique isotopy class of internal simple curves in $\mathcal{B}_\infty$ joining $\mathfrak{p}$ and $\mathfrak{q}$, which is called the *segment* of endpoints $\mathfrak{p}$, $\mathfrak{q}$; and is written as $[\mathfrak{p}, \mathfrak{q}]$ or $[\mathfrak{q}, \mathfrak{p}]$. A segment $[\mathfrak{p}, \mathfrak{q}]$ is called an *edge* if $\{\mathfrak{p}, \mathfrak{q}\} = \{\mathfrak{l}_i, \mathfrak{l}_{i+1}\}$ or $\{\mathfrak{p}, \mathfrak{q}\} = \{\mathfrak{r}_i, \mathfrak{r}_{i+1}\}$ for some $i \in \mathbb{Z}$; and otherwise, an *arc*. More explicitly, an arc in $\mathcal{B}_\infty$ is a segment of the form $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $|i - j| > 1$ called an *upper arc*, or $[\mathfrak{r}_i, \mathfrak{r}_j]$ with $|i - j| > 1$ called a *lower arc*, or $[\mathfrak{l}_i, \mathfrak{r}_j]$ with $i, j \in \mathbb{Z}$ called a *connecting arc*. We shall denote by $\operatorname{arc}(\mathcal{B}_\infty)$ the set of arcs in $\mathcal{B}_\infty$, which is equipped with a translation τ as defined below.

3.1. Definition. (1) For a marked point $\mathfrak{p}$ in $\mathcal{B}_\infty$, we define its *translate* $\tau\mathfrak{p}$ to be $\mathfrak{l}_{i+1}$ if $\mathfrak{p} = \mathfrak{l}_i$; and $\mathfrak{r}_{i+1}$ if $\mathfrak{p} = \mathfrak{r}_i$.
(2) For an arc $u = [\mathfrak{p}, \mathfrak{q}]$ in $\mathcal{B}_\infty$, we define its *translate* τu to be the arc $[\tau\mathfrak{p}, \tau\mathfrak{q}]$.

Remark. The translation τ is a permutation on the marked points and on the arcs. Its inverse will be written as τ^-.

One says that two arcs u, v *cross*, or (u, v) is a *crossing pair*, if every curve in u crosses each of the curves in v. Clearly, an arc does not cross itself, two crossing arcs do not share a common endpoint, and an upper arc does not cross any lower arc. The following easy observation will be frequently used without a reference.

3.2. Lemma. *Let (u, v) be a crossing pair of arcs in $\mathcal{B}_\infty$.*

(1) *If $u = [\mathfrak{l}_i, \mathfrak{l}_j]$ with $i < j$, then $v = [\mathfrak{l}_p, \mathfrak{r}_q]$ with $i < p < j$; or $v = [\mathfrak{l}_p, \mathfrak{l}_q]$ with $i < p < j < q$ or $p < i < q < j$.*
(2) *If $u = [\mathfrak{r}_i, \mathfrak{r}_j]$ with $i > j$, then $v = [\mathfrak{l}_p, \mathfrak{r}_q]$ with $i > q > j$; or $v = [\mathfrak{r}_p, \mathfrak{r}_q]$ with $i > p > j > q$ or $p > i > q > j$.*
(3) *If $u = [\mathfrak{l}_i, \mathfrak{r}_j]$, then $v = [\mathfrak{l}_p, \mathfrak{l}_q]$ with $p < i < q$; or $v = [\mathfrak{r}_p, \mathfrak{r}_q]$ with $p > j > q$; or $v = [\mathfrak{l}_p, \mathfrak{r}_q]$ with $i > p$ and $j > q$ or $i < p$ and $j < q$.*

Remark. By Lemma 3.2, a pair of arcs (u, v) is crossing if and only if so is $(\tau u, \tau v)$. Moreover, an arc u crosses both τu and $\tau^- u$.

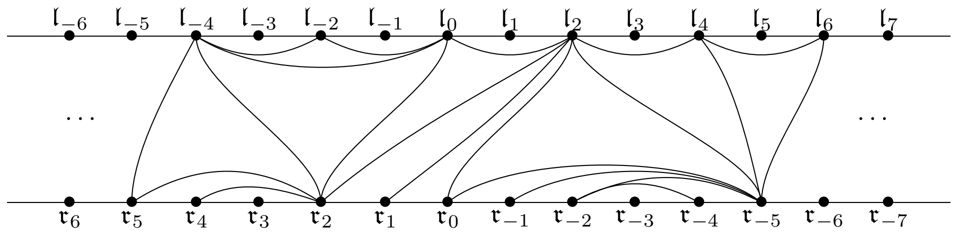
The connecting arcs in $\mathcal{B}_\infty$ play a special role in our investigation.

3.3. Lemma. *The set of connecting arcs in $\mathcal{B}_\infty$ is partially ordered in such a way that $[\mathfrak{l}_i, \mathfrak{r}_j] \le [\mathfrak{l}_r, \mathfrak{r}_s]$ if and only if $i \le r$ and $j \ge s$. In particular, two connecting arcs are comparable if and only if they do not cross.*

The following notion is our main objective of study in this section.

3.4. Definition. A maximal set $\mathbb{T}$ of pairwise non-crossing arcs in $\mathcal{B}_\infty$ is called a *triangulation* of $\mathcal{B}_\infty$.

Example. The following picture shows a triangulation of $\mathcal{B}_\infty$:



We shall study some properties of connecting arcs of a triangulation. Given a triangulation $\mathbb{T}$ of $\mathcal{B}_\infty$, we denote by $C(\mathbb{T})$ the set of connecting arcs of $\mathbb{T}$.

3.5. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$, and let p be an integer.*

(1) *If there exist infinitely many $i < p$ such that $[\mathfrak{l}_i, \mathfrak{l}_{j_i}] \in \mathbb{T}$ for some $j_i \geq p$ or infinitely many $j > -p$ such that $[\mathfrak{l}_{i_j}, \mathfrak{r}_j] \in \mathbb{T}$ for some $i_j \geq p$, then no $\mathfrak{l}_i$ with $i < p$ is an endpoint of an arc of $C(\mathbb{T})$.*

(2) *If there exist infinitely many $i > p$ such that $[\mathfrak{l}_{j_i}, \mathfrak{l}_i] \in \mathbb{T}$ for some $j_i \leq p$ or infinitely many $j < -p$ such that $[\mathfrak{l}_{i_j}, \mathfrak{r}_j] \in \mathbb{T}$ for some $i_j \leq p$, then no $\mathfrak{l}_i$ with $i > p$ is an endpoint of an arc of $C(\mathbb{T})$.*

Proof. We shall prove only Statement (1). Consider a connecting arc $v = [\mathfrak{l}_r, \mathfrak{r}_s]$ with $r < p$. If the first situation in Statement (1) occurs, then there exists some integer $i < r$ such that $[\mathfrak{l}_i, \mathfrak{l}_{j_i}] \in \mathbb{T}$ for some $j_i \geq p$. In this case, v crosses $[\mathfrak{l}_i, \mathfrak{l}_{j_i}]$, and hence, $v \notin \mathbb{T}$. If the second situation occurs, then there exists some $j > s$ such that $[\mathfrak{l}_{i_j}, \mathfrak{r}_j] \in \mathbb{T}$ for some $i_j \geq p$. In this case, v crosses $[\mathfrak{l}_{i_j}, \mathfrak{r}_j]$, and hence, $v \notin \mathbb{T}$. The proof of the lemma is completed.

Remark. A similar statement holds for lower marked points.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. For each arc u in $\mathcal{B}_\infty$, we shall denote by $\mathbb{T}_u$ the set of arcs of $\mathbb{T}$ crossing u.

3.6. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$ containing connecting arcs, and let u be an arc in $\mathcal{B}_\infty$. If $\mathbb{T}_u$ is infinite, then some marked point in $\mathcal{B}_\infty$ is an endpoint of infinitely many arcs in $\mathbb{T}_u$.*

Proof. Assume that $\mathbb{T}_u$ is infinite. If u is not a connecting arc, then the lemma is evident. Suppose that u is a connecting arc. Choose $v \in C(\mathbb{T})$. Clearly, $u \neq v$. If u, v do not cross, then they enclose a bounded region of $\mathcal{B}_\infty$ having only finitely many marked points. Then each arc in $\mathbb{T}_u$ has an endpoint in the enclosed region. So the lemma holds in this case. If u, v crosses, then they enclose two bounded regions of $\mathcal{B}_\infty$, each having only finitely many marked points. Again, each arc in $\mathbb{T}_u$ has an endpoint in one of these regions. The proof of the lemma is completed.

An upper marked point $\mathfrak{l}_i$ in $\mathcal{B}_\infty$ is said to be *covered* by an upper arc $[\mathfrak{l}_r, \mathfrak{l}_s]$ if $r < i < s$; and a lower marked point $\mathfrak{r}_j$ is *covered* by a lower arc $[\mathfrak{r}_p, \mathfrak{r}_q]$ if $p > j > q$.

3.7. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. If $C(\mathbb{T})$ is empty, then one of the following situations occurs.*

(1) *Every upper marked point in $\mathcal{B}_\infty$ is an endpoint of at most finitely many upper arcs of $\mathbb{T}$ and covered by infinitely many upper arcs of $\mathbb{T}$.*

(2) *Every lower marked point in $\mathcal{B}_\infty$ is is an endpoint of at most finitely many lower arcs of $\mathbb{T}$ and covered by infinitely many lower arcs of $\mathbb{T}$.*

Proof. Assume that neither of the two statements holds. We claim that some upper marked point $\mathfrak{l}_p$ is not covered by any upper arc of $\mathbb{T}$. If some upper marked point $\mathfrak{l}_s$ is an endpoint of infinitely many upper arcs of $\mathbb{T}$, since the arcs in $\mathbb{T}$ do not cross each other, $\mathfrak{l}_s$ is not covered by any upper arc of $\mathbb{T}$. Otherwise, since Statement (1) does not hold, some upper marked point $\mathfrak{l}_t$ is covered only by a finite set S of upper arcs of $\mathbb{T}$. We may assume that S is non-empty. Let p be minimal for which $\mathfrak{l}_p$ is an endpoint of an arc in S. Using again the fact that the arcs in $\mathbb{T}$ do not cross each other, we see that $\mathfrak{l}_p$ is not covered by any upper arc of $\mathbb{T}$. This establishes our claim. Similarly, there exists a lower marked point $\mathfrak{r}_q$ which is not covered by any lower arc of $\mathbb{T}$. If $C(\mathbb{T}) = \emptyset$, then $[\mathfrak{l}_p, \mathfrak{r}_q]$ does not belong to $\mathbb{T}$ and does not cross any of the arcs of $\mathbb{T}$, a contradiction. The proof of the lemma is completed.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. An upper marked point $\mathfrak{l}_p$ is called *left $\mathbb{T}$-bounded* if $[\mathfrak{l}_i, \mathfrak{l}_p], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for at most finitely many $i < p$ and at most finitely many $j > -p$; and *left $\mathbb{T}$-unbounded* if $[\mathfrak{l}_i, \mathfrak{l}_p], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for infinitely many $i < p$ and infinitely many $j > -p$. Moreover, $\mathfrak{l}_p$ is called *right $\mathbb{T}$-bounded* if $[\mathfrak{l}_p, \mathfrak{l}_i], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for at most finitely many $i > p$ and at most finitely many $j < -p$; and *right $\mathbb{T}$-unbounded* if $[\mathfrak{l}_p, \mathfrak{l}_i], [\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for infinitely many $i > p$ and infinitely many $j < -p$. In a similar manner, we shall define a lower marked point to be *left $\mathbb{T}$-bounded*, *left $\mathbb{T}$-unbounded*, *right $\mathbb{T}$-bounded*, and *right $\mathbb{T}$-unbounded.* Note that, in these definitions, not bounded does not mean unbounded.

3.8. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$ with $[\mathfrak{l}_p, \mathfrak{r}_q] \in C(\mathbb{T})$.*

(1) *If $\mathfrak{l}_p$ is left (respectively, right) $\mathbb{T}$-bounded, then some $\mathfrak{l}_i$ with $i < p$ (respectively, $i > p$) is an endpoint of an arc of $C(\mathbb{T})$.*

(2) *If $\mathfrak{r}_q$ is left (respectively, right) $\mathbb{T}$-bounded, then some $\mathfrak{r}_j$ with $j > q$ (respectively, $j < q$) is an endpoint of an arc of $C(\mathbb{T})$.*

Proof. We shall prove only the first part of Statement (1). Assume that no $\mathfrak{l}_i$ with $i < p$ is an endpoint of any arc of $C(\mathbb{T})$ and $\mathfrak{l}_p$ is left $\mathbb{T}$-bounded. Then there exists at most finitely many $i < p-1$ such that $[l_i, l_p] \in \mathbb{T}$ and we may suppose that q is maximal such that $u = [\mathfrak{l}_p, \mathfrak{r}_q] \in C(\mathbb{T})$. Define $r = p-1$ if $[\mathfrak{l}_i, \mathfrak{l}_p] \notin \mathbb{T}$ for every $i < p-1$; and otherwise, let $r < p-1$ be minimal such that $[\mathfrak{l}_r, \mathfrak{l}_p] \in \mathbb{T}$. By the first part of the assumption, $v = [\mathfrak{l}_r, \mathfrak{r}_q] \notin \mathbb{T}$. Hence, v crosses some arc w of $\mathbb{T}$.

Since w does not cross u, it is not a lower arc. If w is a connecting arc, using again the assumption, we obtain $w = [\mathfrak{l}_p, \mathfrak{r}_m]$ with $m > q$, contrary to the maximality of q. Hence, $w = [\mathfrak{l}_s, \mathfrak{l}_t]$ with $s < r < t \le p$. If $t = p$, then $r \le s$ by definition, a contradiction. If $t < p$, then $r < p-1$, and by definition, $[\mathfrak{l}_r, \mathfrak{l}_p] \in \mathbb{T}$ which crosses w, a contradiction. The proof of the lemma is completed.

Let $\varSigma$ be a set of arcs in $\mathcal{B}_\infty$. We shall denote by $\tau\varSigma$ the set of arcs of the form τu with $u \in \varSigma$; and by $\tau^-\varSigma$ the set of arcs of the form $\tau^- v$ with $v \in \varSigma$.

3.9. Definition. A set Ω of arcs in $\mathcal{B}_\infty$ is called *compact* if it admits a finite subset Σ such that every arc in Ω crosses some arc of $\tau\Sigma$ as well as some arc of $\tau^-\Sigma$.

Since every arc u crosses τu and $\tau^- u$, a finite subset of arc$(\mathcal{B}_\infty)$ is compact by definition. A subset of a set is called *co-finite* if its complement is finite.

3.10. Lemma. *Let Ω be a set of arcs in $\mathcal{B}_\infty$. If Ω has a compact co-finite subset, then Ω is compact.*

Proof. Assume that Ω has a compact co-finite subset Θ, with Σ a finite subset of Θ satisfying the condition stated in Definition 3.9. Let Λ be the union of Σ and the complement of Θ in Ω, which is finite by the assumption. In particular, $\tau\Sigma \subseteq \tau\Lambda$ and $\tau^-\Sigma \subseteq \tau^-\Lambda$. Let u be an arc in $\mathcal{B}_\infty$. If $u \in \Theta$, then it crosses some arc of $\tau\Sigma$ and some arc of $\tau^-\Sigma$. Otherwise, $u \in \Lambda$, which crosses both τu and $\tau^- u$. The proof of the lemma is completed.

The following notion is essential for describing the cluster-tilting subcategories of a cluster category of type $\mathbb{A}_\infty^\infty$ in the next section.

3.11. Definition. A triangulation $\mathbb{T}$ of $\mathcal{B}_\infty$ is called *compact* if $\mathbb{T}_u$ is compact for every arc u in $\mathcal{B}_\infty$.

The rest of this section is devoted to finding a criterion for a triangulation of $\mathcal{B}_\infty$ to be compact. We start with some properties of a compact triangulation.

3.12. Lemma. *Let $\mathbb{T}$ be a compact triangulation of $\mathcal{B}_\infty$, and let p, q be integers.*

(1) *If $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}$ for infinitely many $i < p$ (respectively, $i > p$), then $\mathfrak{l}_p$ is left (respectively, right) $\mathbb{T}$-unbounded.*
(2) *If $[\mathfrak{r}_j, \mathfrak{r}_q] \in \mathbb{T}$ for infinitely many $j > q$ (respectively, $j < q$), then $\mathfrak{r}_q$ is left (respectively, right) $\mathbb{T}$-unbounded.*

Proof. We shall prove only the first part of Statement (1). Assume that $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}$ for infinitely many $i < p$. We shall need to show that $[\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for infinitely many $j > -p$. Suppose that this is not the case. Then, there exists an integer q such that $[\mathfrak{l}_p, \mathfrak{r}_j] \notin \mathbb{T}$ for all $j > q$. Consider the connecting arc $u = [\mathfrak{l}_{p-1}, \mathfrak{r}_q]$. By the assumption, $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}_u$ for infinitely many $i < p-1$. Being compact, $\mathbb{T}_u$ has a finite subset Σ satisfying the condition stated in Definition 3.9. Observe that there exists an integer $t < p-1$ such that $[\mathfrak{l}_j, \mathfrak{l}_p] \notin \Sigma$ for all $j < t$. Moreover, $w = [\mathfrak{l}_r, \mathfrak{l}_p] \in \mathbb{T}_u$ for some $r < t$.

We claim that w does not cross $\tau^- v$ for any $v \in \Sigma$. Indeed, this is trivial if v is a lower arc in Σ. Assume that v is a connecting arc in Σ. Then $v = [\mathfrak{l}_m, \mathfrak{r}_n]$ with $m > p-1$ and $n > q$, or else, $m < p-1$ and $n < q$. Since v does not cross any of the infinitely many arcs $[\mathfrak{l}_i, \mathfrak{l}_p]$ of $\mathbb{T}_u$ with $i < p-1$, we see that $m > p-1$ and $n > q$. By the assumption on q, we obtain $m > p$. Since $w = [\mathfrak{l}_r, \mathfrak{l}_p]$ with $p \le m-1$, it does not cross $\tau^- v = [\mathfrak{l}_{m-1}, \mathfrak{r}_{n-1}]$.

Suppose now that v is an upper arc in Σ. Then $v = [\mathfrak{l}_m, \mathfrak{l}_n]$ with $m < p-1 < n$. Since v does not cross any of the infinitely many arcs $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}_u$ with $i < p$, we have $n = p$, that is, $v = [\mathfrak{l}_m, \mathfrak{l}_p]$ with $m < p-1$. By the assumption on t, we obtain $t \le m$. Since $w = [\mathfrak{l}_r, \mathfrak{l}_p]$ with $r \le m-1$, it does not cross $\tau^- v = [\mathfrak{l}_{m-1}, \mathfrak{l}_{p-1}]$. This establishes our claim, a contradiction. The proof of the lemma is completed.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. It follows from Lemma 3.3 that $C(\mathbb{T})$ is well ordered whenever it is not empty.

3.13. PROPOSITION. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. If $\mathbb{T}$ is compact, then $C(\mathbb{T})$ is a double-infinite chain.*

Proof. Let $\mathbb{T}$ be compact. Suppose that $C(\mathbb{T})$ is empty. By Lemma 3.7, we may assume that every upper marked point is an endpoint of at most finitely many upper arcs of $\mathbb{T}$ and covered by infinitely many upper arcs of $\mathbb{T}$. Consider the arc $u_0 = [\mathfrak{l}_0, \mathfrak{r}_0]$. Since every upper arc covering $\mathfrak{l}_0$ crosses u_0, the set $U(\mathbb{T}_{u_0})$ of upper arcs of $\mathbb{T}_{u_0}$ is infinite. Being compact, $\mathbb{T}_{u_0}$ has a finite subset Σ satisfying the condition stated in Definition 3.9. There exist r_0, s_0 such that no $\mathfrak{l}_i$ with $i < r_0$ or $i > s_0$ is an endpoint of any arc of Σ. Since each upper marked point is an endpoint of at most finitely many arcs of $U(\mathbb{T}_{u_0})$, the infinite set $U(\mathbb{T}_{u_0})$ contains an arc $u_1 = [\mathfrak{l}_{r_1}, \mathfrak{l}_{s_1}]$ with $r_1 < r_0 - 1$ and $s_1 > s_0 + 1$. Let $v \in \Sigma$. By the assumption on r_0, s_0, either $v = [\mathfrak{l}_r, \mathfrak{l}_s]$ with $r_0 \le r < s \le s_0$ or v is a lower arc. In either case, u_1 does not cross $\tau^- v$ or τv, a contradiction. This shows that $C(\mathbb{T}) \neq \emptyset$.

By Lemma 3.3, $C(\mathbb{T})$ is well ordered. Suppose that $C(\mathbb{T})$ has a minimal element $[\mathfrak{l}_p, \mathfrak{r}_q]$. Since arc($\mathbb{T}$) contains no crossing pair, we deduce from the minimality of $[\mathfrak{l}_p, \mathfrak{r}_q]$ that no $\mathfrak{l}_i$ with $i < p$ is an endpoint of an arc of $C(\mathbb{T})$. By Lemma 3.8(1), $\mathfrak{l}_p$ is not left $\mathbb{T}$-bounded, and by Lemma 3.12(1), $\mathfrak{l}_p$ is left $\mathbb{T}$-unbounded. In particular, $[\mathfrak{l}_p, \mathfrak{r}_j] \in \mathbb{T}$ for some $j > q$, contrary to the minimality of $[\mathfrak{l}_p, \mathfrak{r}_q]$. Similarly, one can show that $C(\mathbb{T})$ has no maximal element. Since every interval in $C(\mathbb{T})$ is evidently finite, $C(\mathbb{T})$ is a double infinite chain. The proof of the proposition is completed.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. A marked point $\mathfrak{p}$ in $\mathcal{B}_\infty$ is called a *left* $\mathbb{T}$-*fountain base* if $\mathfrak{p}$ is left $\mathbb{T}$-unbounded but right $\mathbb{T}$-bounded. In this case, if $\mathfrak{p} = \mathfrak{l}_p$, then the set of arcs in $\mathbb{T}$ of the form $[\mathfrak{l}_i, \mathfrak{l}_p]$ with $i < p - 1$ or $[\mathfrak{l}_p, \mathfrak{r}_j]$ with $j > -p$ is called a *left fountain* of $\mathbb{T}$ at $\mathfrak{p}$; and if $\mathfrak{p} = \mathfrak{r}_q$, then the set of arcs in $\mathbb{T}$ of the form $[\mathfrak{r}_i, \mathfrak{r}_q]$ with $i > q + 1$ or $[\mathfrak{l}_j, \mathfrak{r}_q]$ with $j < -q$ is called a *left fountain* of $\mathbb{T}$ at $\mathfrak{p}$. In a dual fashion, we define a *right* $\mathbb{T}$-*fountain base* and a *right fountain* of $\mathbb{T}$ at a right fountain base. Further, a marked point $\mathfrak{p}$ is called a *full* $\mathbb{T}$-*fountain base* if $\mathfrak{p}$ is left and right $\mathbb{T}$-unbounded; and in this case, the set of arcs of $\mathbb{T}$ having $\mathfrak{p}$ as an endpoint is called a *full fountain* of $\mathbb{T}$ at $\mathfrak{p}$. For brevity, a left, right or full $\mathbb{T}$-fountain base $\mathfrak{p}$ will be simply called a $\mathbb{T}$-*fountain base*; and the left, right or full fountain at $\mathfrak{p}$ will be simply called the *fountain* at $\mathfrak{p}$ and denoted by $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$.

3.14. LEMMA. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$, containing at least one fountain.*

(1) *If $\mathfrak{p}$ is a full $\mathbb{T}$-fountain base, then it is the unique $\mathbb{T}$-fountain base and it is an endpoint of all connecting arcs of $\mathbb{T}$.*
(2) *If $\mathfrak{p}$, $\mathfrak{q}$ are two distinct $\mathbb{T}$-fountain bases, then they are the only $\mathbb{T}$-fountain bases with one being a left $\mathbb{T}$-fountain base and the other one being a right $\mathbb{T}$-fountain base.*

Proof. Assume that some upper marked point $\mathfrak{l}_p$ is left $\mathbb{T}$-unbounded. We claim that $\mathfrak{l}_p$ is the only left $\mathbb{T}$-unbounded marked point in $\mathcal{B}_\infty$ and none of the $\mathfrak{l}_i$ with $i < p$ is an endpoint of some connecting arc of $\mathbb{T}$. Indeed, the second part of the claim follows from Lemma 3.5(1). As a consequence, the $\mathfrak{l}_i$ with $i < p$ and the $\mathfrak{r}_j$ with $j \in \mathbb{Z}$ are not left $\mathbb{T}$-unbounded. Since $\mathfrak{p}$ is a $\mathbb{T}$-fountain base, $\mathbb{T}$ contains a connecting arc $[\mathfrak{l}_p, \mathfrak{r}_q]$. Since arc($\mathbb{T}$) contains no crossing pair, $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $i < p < j$

does not belong to $\mathbb{T}$. In particular, no $\mathfrak{l}_j$ with $j > p$ is left $\mathbb{T}$-unbounded. This establishes our claim.

Suppose now that $\mathfrak{p}$ is a full $\mathbb{T}$-fountain base. We shall consider only the case where $\mathfrak{p}$ is an upper marked point, say $\mathfrak{p} = \mathfrak{l}_p$. By our claim and its dual, $\mathfrak{p}$ is the only $\mathbb{T}$-fountain base. Moreover, no $\mathfrak{l}_i$ with $i \neq p$ is an end-point of some connecting arc of $\mathbb{T}$. Thus, $\mathfrak{p}$ is an endpoint of all connecting arcs of $\mathbb{T}$. This establishes Statement (1). Finally, Statement (2) follows from the first part of the claim and its dual. The proof of the lemma is completed.

3.15. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$, and let v be an arc in $\mathcal{B}_\infty$. If v crosses infinitely many arcs of a full fountain of $\mathbb{T}$, then $\mathbb{T}_v$ is compact.*

Proof. Assume that v crosses infinitely many arcs of a full fountain $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ of $\mathbb{T}$. We shall consider only the case where $\mathfrak{p}$ is an upper marked point, say $\mathfrak{p} = \mathfrak{l}_p$ for some $p \in \mathbb{Z}$. Then, v is evidently not a lower arc.

Suppose that v is an upper arc. Then $v = [\mathfrak{l}_r, \mathfrak{l}_s]$ with $r < p < s$. Let i_0 with $i_0 < r$ be maximal such that $v_1 = [\mathfrak{l}_{i_0}, \mathfrak{l}_p] \in \mathbb{T}$, and let j_0 with $j_0 > s$ be minimal such that $v_2 = [\mathfrak{l}_p, \mathfrak{l}_{j_0}] \in \mathbb{T}$. We claim that $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is co-finite in $\mathbb{T}_v$. Indeed, let u be an arc in $\mathbb{T}_v$ but not in $\mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. Then u is not a lower arc, and by Lemma 3.14(1), it is an upper arc. Since u does not cross v_1 or v_2, we see that $u = [\mathfrak{l}_i, \mathfrak{l}_j]$ with $i_0 \le i < r < j < p$ or $p < i < s < j \le j_0$. Therefore, our claim holds. In order to prove that $\mathbb{T}_v$ is compact, by Lemma 3.10, it suffices to show that $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$ is compact. Note that $v_1, v_2 \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$ with $\tau v_1 = [\mathfrak{l}_{i_0+1}, \mathfrak{l}_{p+1}]$ and $\tau^- v_2 = [\mathfrak{l}_{p-1}, \mathfrak{l}_{j_0-1}]$. Let $w \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. If w is an upper arc, then we deduce from the maximality of i_0 and the minimality of j_0 that $w = [\mathfrak{l}_m, \mathfrak{l}_p]$ with $m \le i_0$ or $w = [\mathfrak{l}_p, \mathfrak{l}_n]$ with $j_0 \le n$. In the first situation, since $m < i_0 + 1 < r + 1 \le p < p + 1$ and $m < r \le p - 1 < p < s \le j_0 - 1$, we see that w crosses both τv_1 and $\tau^- v_2$. In the second situation, since $i_0 + 1 \le r < p < p + 1 \le s < n$ and $p < s \le j_0 - 1 < n$, we see that w crosses both τv_1 and $\tau^- v_2$. Hence, $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$ is indeed compact. In a similar way, one can deal with the case where v is a connecting arc. The proof of the lemma is completed.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. For a marked point $\mathfrak{p}$, we shall denote by $\mathbb{E}_{\mathbb{T}}(\mathfrak{p})$ the set of arcs of $\mathbb{T}$ having $\mathfrak{p}$ as an endpoint. If $\mathfrak{p}$ is a $\mathbb{T}$-fountain base, then the $\mathbb{T}$-fountain $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is by definition a co-finite subset of $\mathbb{E}_{\mathbb{T}}(\mathfrak{p})$.

3.16. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$ with $\mathfrak{p}$ a left or right $\mathbb{T}$-fountain base, and let v be an arc in $\mathcal{B}_\infty$. If v crosses infinitely many arcs of $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$, then $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is compact and co-finite in $\mathbb{E}_{\mathbb{T}}(\mathfrak{p})$.*

Proof. We shall consider only the case where $\mathfrak{p}$ is a left $\mathbb{T}$-fountain base and $\mathfrak{p} = \mathfrak{l}_p$ for some $p \in \mathbb{Z}$. Assume that v crosses infinitely many arcs of $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$. Then v is not a lower arc, since every lower arc crosses at most finitely many arcs of $\mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. Since $\mathfrak{l}_p$ is right $\mathbb{T}$-bounded, one of the endpoints of v is $\mathfrak{l}_r$ with $r < p$. That is, $v = [\mathfrak{l}_r, \mathfrak{l}_s]$ with $r < p < s$ or $v = [\mathfrak{l}_r, \mathfrak{r}_s]$ with $r < p$ and $s \in \mathbb{Z}$.

Let $w \in \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$, which does not cross v. If $v = [\mathfrak{l}_r, \mathfrak{l}_s]$, then $w = [\mathfrak{l}_j, \mathfrak{l}_p]$ with $r \le j < p-1$. If $v = [\mathfrak{l}_r, \mathfrak{r}_s]$, then $w = [\mathfrak{l}_j, \mathfrak{l}_p]$ with $r \le j < p-1$ or $w = [\mathfrak{l}_p, \mathfrak{r}_t]$ with $-p < t \le s$. Thus, $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is co-finite in $\mathbb{F}_{\mathbb{T}}(\mathfrak{p})$, and then, co-finite in $\mathbb{E}_{\mathbb{T}}(\mathfrak{p})$.

We shall show that $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is compact. Since $\mathfrak{l}_p$ is left $\mathbb{T}$-unbounded, there exists a maximal $m\,(< r)$ such that $v_1 = [\mathfrak{l}_m, \mathfrak{l}_p] \in \mathbb{T}$. Clearly, $v_1 \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. Let $u \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. If u is a connecting arc, then $u = [\mathfrak{l}_p, \mathfrak{r}_t]$ for some $t > -p$, which crosses $\tau v_1 = [\mathfrak{l}_{m+1}, \mathfrak{l}_{p+1}]$. Otherwise, $u = [\mathfrak{l}_t, \mathfrak{l}_p]$ with $t < r$. By the maximality of m, we obtain $t \le m$, Therefore, u crosses $\tau v_1 = [\mathfrak{l}_{m+1}, \mathfrak{l}_{p+1}]$.

Next, in case $v = [\mathfrak{l}_r, \mathfrak{l}_s]$, let $n > -p$ be minimal such that $[\mathfrak{l}_p, \mathfrak{r}_n] \in \mathbb{T}$; and in case $v = [\mathfrak{l}_r, \mathfrak{r}_s]$, let $n > \max\{-p, s\}$ be minimal such that $[\mathfrak{l}_p, \mathfrak{r}_n] \in \mathbb{T}$. In either case, set $v_2 = [\mathfrak{l}_p, \mathfrak{r}_n]$. Clearly $v_2 \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. Let $u \in \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{l}_p)$. If u is an upper arc, then $u = [\mathfrak{l}_t, \mathfrak{l}_p]$ with $t < r$, which crosses $\tau^- v_2 = [\mathfrak{l}_{p-1}, \mathfrak{r}_{n-1}]$. Otherwise, $u = [\mathfrak{l}_p, \mathfrak{r}_t]$, where $t > -p$, and $t > s$ in case $v = [\mathfrak{l}_r, \mathfrak{r}_s]$. By the minimality of n, we obtain $t \ge n$. Hence, u crosses $\tau^- v_2 = [\mathfrak{l}_{p-1}, \mathfrak{r}_{n-1}]$. This shows that $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p})$ is compact. The proof of the lemma is completed.

Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$. A marked point in $\mathcal{B}_\infty$ is said to be $\mathbb{T}$-*bounded* if it is both left and right $\mathbb{T}$-bounded, or equivalently, it is an endpoint of at most finitely many arcs of $\mathbb{T}$.

3.17. Lemma. *Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$ such that every marked point in $\mathcal{B}_\infty$ is either $\mathbb{T}$-bounded or an endpoint of infinitely many connecting arcs of $\mathbb{T}$. Then every marked point in $\mathcal{B}_\infty$ is either $\mathbb{T}$-bounded or a $\mathbb{T}$-fountain base.*

Proof. Let $\mathfrak{p}$ be a marked point, which is an endpoint of infinitely many arcs in $C(\mathbb{T})$. It suffices to show that $\mathfrak{p}$ is a $\mathbb{T}$-fountain base. Let $C_\mathfrak{p}(\mathbb{T})$ denote the arcs of $C(\mathbb{T})$ having $\mathfrak{p}$ as an endpoint. We shall consider only the case where $\mathfrak{p} = \mathfrak{l}_p$ for some $p \in \mathbb{Z}$. Being infinite and well-ordered, $C_\mathfrak{p}(\mathbb{T})$ has no minimal element or no maximal element. We may assume that the first case occurs.

We claim that $\mathfrak{p}$ is left $\mathbb{T}$-unbounded. Indeed, having no minimal element, $[\mathfrak{l}_p, \mathfrak{r}_j] \in C_\mathfrak{p}(\mathbb{T})$ for infinitely many $j > -p$. Since arc($\mathbb{T}$) contains no crossing pair, no $\mathfrak{l}_i$ with $i < p$ is an endpoint of a connecting arc of $\mathbb{T}$. By the assumption stated in the lemma, $\mathfrak{l}_i$ with $i < p$ is $\mathbb{T}$-bounded. Suppose that the claim is false. Then $[\mathfrak{l}_i, \mathfrak{l}_p] \in \mathbb{T}$ for at most finitely many $i < p$. Define $s = p - 1$ if $[\mathfrak{l}_j, \mathfrak{l}_p] \notin \mathbb{T}$ for every $j < p - 1$; and otherwise, let $s < p - 1$ be minimal such that $[\mathfrak{l}_s, \mathfrak{l}_p] \in \mathbb{T}$. Since $\mathfrak{l}_s$ is an endpoint of at most finitely many arcs of $\mathbb{T}$, we may define $t = s - 1$ if $[\mathfrak{l}_i, \mathfrak{l}_s] \notin \mathbb{T}$ for every $i < s - 1$; and otherwise, let $t < s - 1$ be minimal such that $[\mathfrak{l}_t, \mathfrak{l}_s] \in \mathbb{T}$. Consider the upper arc $v = [\mathfrak{l}_t, \mathfrak{l}_p] \notin \mathbb{T}$. Observe that v does not cross any arc in $C(\mathbb{T})$. Therefore, v crosses some upper arc u of $\mathbb{T}$. Since u does not cross any arc of $C_\mathfrak{p}(\mathbb{T})$, we obtain $u = [\mathfrak{l}_{t_1}, \mathfrak{l}_{s_1}]$ with $t_1 < t < s_1 < p$. If $s < s_1$, then $s < p - 1$, and hence, $[\mathfrak{l}_s, \mathfrak{l}_p]$ lies in $\mathbb{T}$ and crosses u, a contradiction. If $s_1 < s$, then $t < s - 1$, and hence, $[\mathfrak{l}_t, \mathfrak{l}_s] \in \mathbb{T}$ which crosses u, a contradiction. Thus, $s_1 = s$, a contradiction to the definition of t. This establishes our claim.

If $C_\mathfrak{p}(\mathbb{T})$ has no maximal element, a dual argument shows that $\mathfrak{p}$ is right $\mathbb{T}$-unbounded, and hence, it is a full $\mathbb{T}$-fountain base. Assume that $C_\mathfrak{p}(\mathbb{T})$ has a maximal element $u_0 = [\mathfrak{l}_p, \mathfrak{r}_q]$. If $\mathfrak{p}$ is right $\mathbb{T}$-bounded, then $\mathfrak{l}_p$ is a left $\mathbb{T}$-fountain base. Otherwise, we deduce from the maximality of u_0 that $[\mathfrak{l}_p, \mathfrak{l}_j] \in \mathbb{T}$ for infinitely many $j > p$. Since arc($\mathbb{T}$) contains no crossing pair, u_0 is the only connecting arc of $\mathbb{T}$ having $\mathfrak{r}_q$ as an endpoint, and no $\mathfrak{r}_j$ with $j < q$ is an endpoint of any connecting arc of $\mathbb{T}$. By the assumption stated in the lemma, $\mathfrak{r}_q$ is $\mathbb{T}$-bounded, a contradiction to the second part of Lemma 3.8(2). The proof of the lemma is completed.

We are ready to obtain the criterion for a triangulation to be compact.

3.18. Theorem. *A triangulation $\mathbb{T}$ of $\mathcal{B}_\infty$ is compact if and only if it contains infinitely many connecting arcs, and every marked point in $\mathcal{B}_\infty$ is either $\mathbb{T}$-bounded or an endpoint of infinitely many connecting arcs of $\mathbb{T}$.*

Proof. By Lemma 3.12 and Proposition 3.13, we need only to prove the sufficiency. Let $\mathbb{T}$ be a triangulation of $\mathcal{B}_\infty$ such that $\mathcal{C}(\mathbb{T})$ is non-empty and every marked point in $\mathcal{B}_\infty$ is either $\mathbb{T}$-bounded or an endpoint of infinitely many arcs of $C(\mathbb{T})$.

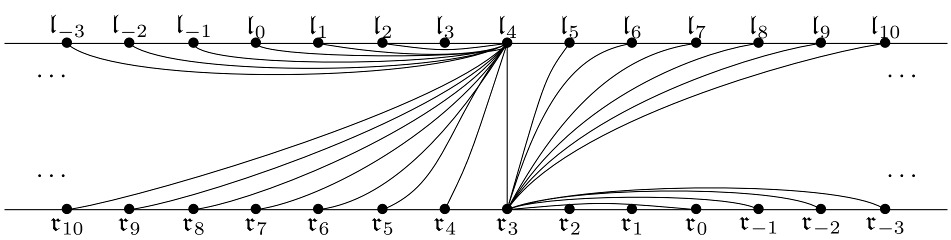
Fix an arc v in $\mathcal{B}_\infty$. We need to show that $\mathbb{T}_v$ is compact. For this purpose, we may assume that $\mathbb{T}_v$ is infinite. By Lemma 3.6, some marked point is an endpoint of infinitely many arcs of $\mathbb{T}_v$; and by Lemma 3.17, such a marked point is a $\mathbb{T}$-fountain base. In view of Lemma 3.14, the number t of such $\mathbb{T}$-fountain bases is at most two. Let $\mathfrak{p}_i$, with $i \in \{1, t\}$, be the $\mathbb{T}$-fountain bases such that $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p}_i)$ is infinite. By Lemma 3.15, we may assume that each $\mathfrak{p}_i$ with $i \in \{1, t\}$ is a left or right $\mathbb{T}$-fountain base; and by Lemma 3.16, each $\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p}_i)$ with $i \in \{1, t\}$ is compact and co-finite in $\mathbb{E}_{\mathbb{T}}(\mathfrak{p}_i)$. It is then easy to see that $\cup_{1\le i\le t}\, \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p}_i)$ is compact. By Lemma 3.10, it suffices to show the claim that $\cup_{1\le i\le t}\, \mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{p}_i)$ is co-finite in $\mathbb{T}_v$. Indeed, given any marked point $\mathfrak{q}$ in $\mathcal{B}_\infty$, we set

$$\Omega(\mathfrak{q}) = \begin{cases} \mathbb{E}_{\mathbb{T}}(\mathfrak{q}) \backslash (\mathbb{T}_v \cap \mathbb{F}_{\mathbb{T}}(\mathfrak{q})), & \text{if } \mathfrak{q} \in \{\mathfrak{p}_1, \mathfrak{p}_t\}; \\ \mathbb{T}_v \cap \mathbb{E}_{\mathbb{T}}(\mathfrak{q}), & \text{if } \mathfrak{q} \notin \{\mathfrak{p}_1, \mathfrak{p}_t\}, \end{cases}$$

which is finite by Lemma 3.16 and the definition of $\{\mathfrak{p}_1, \mathfrak{p}_t\}$. Suppose that v is an upper arc, say $v = [\mathfrak{l}_r, \mathfrak{l}_s]$ with $r < s-1$. Let u be an arc in $\mathbb{T}_v$ but not in $\mathbb{F}_{\mathbb{T}}(\mathfrak{p}_1) \cup \mathbb{F}_{\mathbb{T}}(\mathfrak{p}_t)$. Since u crosses $[\mathfrak{l}_r, \mathfrak{l}_s]$, there exists some $r < i < s$ such that $u \in \mathbb{E}_{\mathbb{T}}(\mathfrak{l}_i)$, and by definition, $u \in \Omega(\mathfrak{l}_i)$. That is, $u \in \cup_{r<i<s}\, \Omega(\mathfrak{l}_i)$. Thus, the claim holds. Similarly, the claim holds in case v is a lower arc.

Suppose that v is a connecting arc, say $v = [\mathfrak{l}_r, \mathfrak{r}_s]$. We consider only the case where $\mathfrak{p}_1$ is an upper marked point and a left $\mathbb{T}$-fountain base. Then $\mathfrak{p}_1 = \mathfrak{l}_{p_1}$ for some $p_1 > r$, and hence, $\mathbb{F}_{\mathbb{T}}(\mathfrak{p}_1)$ contains a connecting arc $w = [\mathfrak{l}_{p_1}, \mathfrak{r}_q]$ with $q > s$. Let u be an arc in $\mathbb{T}_v$ but not in $\cup_{1\le i\le t}\mathbb{F}_{\mathbb{T}}(\mathfrak{p}_i)$. If u is an upper arc then, since it does not cross w, we obtain $u = [\mathfrak{l}_i, \mathfrak{l}_j]$ with $i < r < j \le p_1$. Then, $u \in \Omega(\mathfrak{l}_j)$ for some $r < j \le p_1$. If u is a connecting arc, we deduce from Lemma 3.5(1) that $u = [\mathfrak{l}_i, \mathfrak{r}_j]$ with $i \ge p_1$ and $q \ge j > s$, and hence, $u \in \Omega(\mathfrak{r}_j)$ for some $q \ge j > s$. If u is a lower arc, we obtain $u = [\mathfrak{r}_j, \mathfrak{r}_i]$ with $q \ge j > s$, and hence, $u \in \Omega(\mathfrak{r}_j)$ for some $q \ge j > s$. This establishes the claim. The proof of the theorem is completed.

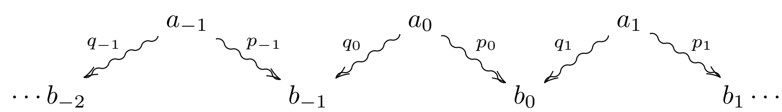
Example. The following shows a compact triangulation of $\mathcal{B}_\infty$ with two fountains.



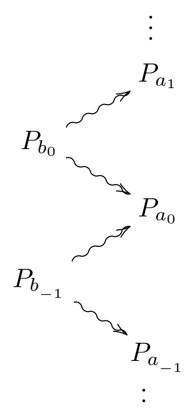
4. Geometric Realization of cluster categories of type $\mathbb{A}_\infty^\infty$

The objective of this section is to study the cluster structure of a cluster category of type $\mathbb{A}_\infty^\infty$ in terms of the triangulations of the infinite strip with marked points $\mathcal{B}_\infty$, as introduced in the previous section.

We start with some algebraic considerations. Let Q denote a quiver of type $\mathbb{A}_\infty^\infty$ with no infinite path, whose vertices are the integers and whose arrows are of the form $n \to (n+1)$ or $n \leftarrow (n+1)$. Let a_i, b_i, $i \in \mathbb{Z}$, be the sources and the sinks, respectively, in Q such that $b_{i-1} < a_i < b_i$. Letting $p_i : a_i \rightsquigarrow b_i$ and $q_i : a_i \rightsquigarrow b_{i-1}$, $i \in \mathbb{Z}$, be the maximal paths, we can picture Q as follows:



By Proposition 2.9, the Auslander-Reiten quiver $\Gamma_{\mathscr{C}(Q)}$ of $\mathscr{C}(Q)$ consists of three connected components, namely, the connecting component $\mathcal{C}_Q$ and two regular components $\mathcal{R}_R$ and $\mathcal{R}_L$. The objects in these components form the fundamental domain $\mathscr{F}(Q)$ of $\mathscr{C}(Q)$. We shall describe the morphisms from objects in $\mathcal{C}_Q$ to those in $\mathcal{R}_R$ or $\mathcal{R}_L$. For this purpose, we need some notation. Observe that $\mathcal{C}_Q$ is of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$ containing a section; see, for definition, [23, (2.1)], as follows:



We denote by R_0 the double infinite sectional path in $\mathcal{C}_Q$ containing the path $P_{b_0} \rightsquigarrow P_{a_0}$, which corresponds to the path $p_0 : a_0 \rightsquigarrow b_0$ in Q; and by L_0 the double infinite sectional path containing the path $P_{b_{-1}} \rightsquigarrow P_{a_0}$, which corresponds to the path $q_0 : a_0 \rightsquigarrow b_{-1}$. Put $R_i = \tau_{\mathscr{C}}^i R_0$ and $L_i = \tau_{\mathscr{C}}^i L_0$, for each $i \in \mathbb{Z}$. Then each object in $\mathcal{C}_Q$ lies in a unique R_i with $i \in \mathbb{Z}$ and in a unique L_j with $j \in \mathbb{Z}$. Recall also that $\mathcal{R}_R, \mathcal{R}_L$ are orthogonal of shape $\mathbb{Z}\mathbb{A}_\infty$ with the string representation $M(p_0)$ being a quasi-simple object in $\mathcal{R}_R$ and $M(q_0)$ being a quasi-simple object in $\mathcal{R}_L$.

4.1. Proposition. *Let M be an object in $\mathcal{C}_Q$. If $i \in \mathbb{Z}$, then*

(1) $M \in R_i$ *if and only if* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, \tau_{\mathscr{C}}^i M(p_0)) \neq 0$; *and in this case, for each* $Y \in \mathcal{R}_R$, *one has* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) \neq 0$ *if and only if* $Y \in \mathcal{W}(\tau_{\mathscr{C}}^i M(p_0))$;

(2) $M \in L_i$ *if and only if* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, \tau_{\mathscr{C}}^i M(q_0)) \neq 0$; *and in this case, for each* $Y \in \mathcal{R}_L$, *one has* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) \neq 0$ *if and only if* $Y \in \mathcal{W}(\tau_{\mathscr{C}}^i M(q_0))$.

Proof. We prove Statement (1) for $i = 0$. Put $d(M) = \dim_k \mathrm{Hom}_{\mathscr{C}(Q)}(M, M(p_0))$. Then, by Proposition 2.12, $d(M) = 0$ or 1; and $d(M) = \dim_k \mathrm{Hom}_{\mathrm{rep}(Q)}(M, M(p_0))$ in case $M \in \mathcal{P}_Q$; see (2.7). In particular, for $x \in Q_0$, we have $d(P_x) = 1$ if and only if x appears on $p_0 : a_0 \rightsquigarrow b_0$. Let b be the immediate predecessor of b_0 in $q_1 : a_1 \rightsquigarrow b_0$. Then, $\mathcal{C}_Q$ has an arrow $P_{b_0} \to P_b$ with $P_b \in R_{-1}$.

Let M be the immediate successor of P_{b_0} in R_0. Since $M(p_0) \not\cong P_{b_0}$ in $\mathrm{rep}(Q)$, applying $\mathrm{Hom}_{\mathrm{rep}(Q)}(-, M(p_0))$ to the almost split sequence in $\mathrm{rep}(Q)$ starting with P_{b_0} yields a short exact sequence. Thus, $d(P_{b_0}) + d(\tau_{\mathscr{C}}^{-} P_{b_0}) = d(M) + d(P_b)$. Since $d(P_{b_0}) = 1$ and $d(P_b) = 0$, we obtain $d(M) = 1$ and $d(\tau_{\mathscr{C}}^{-} P_{b_0}) = 0$. By induction, we can show that $d(M) = 1$ and $d(\tau_{\mathscr{C}}^{-} M) = 0$ if M is a successor of P_{b_0} in R_0.

Assume that M is the immediate predecessor of P_{b_0} in R_0. Let N be the immediate predecessor of P_b in R_{-1}. Since $\mathrm{Hom}_{\mathscr{C}(Q)}(P_b, M(p_0)) = 0$ and $M(p_0) \not\cong M$ in $\mathscr{C}(Q)$, applying $\mathrm{Hom}_{\mathscr{C}(Q)}(-, M(p_0))$ to the almost split triangle in $\mathscr{C}(Q)$ starting with M, we obtain $\mathrm{Hom}_{\mathscr{C}(Q)}(N \oplus P_{b_0}, M(p_0)) \cong \mathrm{Hom}_{\mathscr{C}(Q)}(M, M(p_0))$. Thus, $d(M) = d(P_{b_0}) + d(N) > 0$. Therefore $d(M) = 1$, and hence, $d(N) = 0$. By induction, we have $d(M) = 1$ and $d(\tau_{\mathscr{C}}^{-} M) = 0$ if M is a predecessor of P_{b_0} in R_0.

Suppose that $d(X) = 1$ for some $X \in R_j$ with $j \neq 0$. Write $X = \tau_{\mathscr{C}}^{j} Y$ for some $Y \in R_0$. This yields $\mathrm{Hom}_{\mathscr{C}(Q)}(Y, \tau_{\mathscr{C}}^{-j} M(p_0)) \neq 0$ and $\mathrm{Hom}_{\mathscr{C}(Q)}(Y, M(p_0)) \neq 0$. Observe that $M(p_0)$ and $\tau_{\mathscr{C}}^{-j} M(p_0) = \tau_{Q}^{-j} M(p_0)$ are distinct quasi-simple objects in $\mathcal{R}_R$. By Proposition 2.4, $Y \notin \mathcal{P}_Q$. Thus, $Y = Z[-1]$ with $Z \in \mathcal{I}_Q$. By Lemma 2.6(3), we obtain $\mathrm{Hom}_{\mathrm{rep}(Q)}(M(p_0), \tau_Q Z) \neq 0$ and $\mathrm{Hom}_{\mathrm{rep}(Q)}(\tau^{-j} M(p_0), \tau_Q Z) \neq 0$, which contradicts the dual of Proposition 2.4.

Let $M \in R_0$ and $Y \in \mathcal{R}_R$. If $M \in \mathcal{P}_Q$, then $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) = \mathrm{Hom}_{\mathrm{rep}(Q)}(M, Y)$ with $\mathrm{Hom}_{\mathrm{rep}(Q)}(M, M(p_0)) \neq 0$. By Lemma 2.6, $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) \neq 0$ if and only if $Y \in \mathcal{W}(M(p_0))$. If $M = N[-1]$ with $N \in \mathcal{I}_Q$, then $\mathrm{Hom}_{\mathrm{rep}(Q)}(M(p_0), \tau_Q N) \neq 0$ and $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) \cong D\mathrm{Hom}_{\mathrm{rep}(Q)}(Y, \tau_Q N)$. Thus, $\mathrm{Hom}_{\mathscr{C}(Q)}(M, Y) \neq 0$ if and only if $Y \in \mathcal{W}(M(p_0))$ by the dual of Lemma 2.6. The proof is completed.

Now, we shall parameterize the indecomposable objects of $\mathscr{C}(Q)$ by the arcs in $\mathcal{B}_\infty$, that is, we shall define a bijection $\varphi : \mathscr{F}(Q) \to \mathrm{arc}(\mathcal{B}_\infty)$. Recall that $\mathscr{F}(Q)$ consists of the objects in $\mathcal{C}_Q$, $\mathcal{R}_R$ and $\mathcal{R}_L$. For each $X \in \mathcal{C}_Q$, there exists a unique pair (i, j) of integers such that $X = L_i \cap R_j$, and we set $\varphi(X) = [\mathfrak{l}_i, \mathfrak{r}_j]$. This defines a bijection from the objects in $\mathcal{C}_Q$ onto the connecting arcs in $\mathcal{B}_\infty$.

Next, consider the quasi-simple object $S_L = \tau_{\mathscr{C}}^{-} M(q_0)$ in $\mathcal{R}_L$. For $i \in \mathbb{Z}$, denote by L_i^+ the ray in $\mathcal{R}_L$ starting with $\tau_{\mathscr{C}}^{i} S_L$, and by L_i^- the coray ending with $\tau_{\mathscr{C}}^{i} S_L$. For each $X \in \mathcal{R}_L$, there exists a unique pair of integers (i, j) with $i \leq j$ such that $X = L_i^- \cap L_j^+$, and we set $\varphi(X) = [\mathfrak{l}_{i-1}, \mathfrak{l}_{j+1}]$. This defines a bijection from the objects in $\mathcal{R}_L$ onto the upper arcs in $\mathcal{B}_\infty$. In this way, the quasi-simple objects in $\mathcal{R}_L$ are those mapped by φ to $[\mathfrak{l}_i, \mathfrak{l}_j]$ with $|i - j| = 2$.

Finally, consider the quasi-simple object $S_R = \tau_{\mathscr{C}}^{-} M(p_0)$ in $\mathcal{R}_R$. For $i \in \mathbb{Z}$, denote by R_i^+ the ray in $\mathcal{R}_R$ starting with $\tau_{\mathscr{C}}^{i} S_R$; and by R_i^- the coray ending with $\tau_{\mathscr{C}}^{i} S_R$. For each object $X \in \mathcal{R}_R$, there exists a unique pair (i, j) of integers with $i \geq j$ such that $Y = R_i^+ \cap R_j^-$, and we set $\varphi_R(X) = [\mathfrak{r}_{i+1}, \mathfrak{r}_{j-1}] \in \mathrm{arc}(\mathcal{B}_\infty)$. This yields a bijection from the objects in $\mathcal{R}_R$ onto the lower arcs in $\mathcal{B}_\infty$. Observe that the quasi-simple objects in $\mathcal{R}_R$ are those mapped by φ to $[\mathfrak{r}_i, \mathfrak{r}_j]$ with $|i - j| = 2$. This concludes the definition of our bijection φ. To simplify the notation, for $X \in \mathscr{F}(Q)$ and $u \in \mathrm{arc}(\mathcal{B}_\infty)$, we shall write $a_X = \varphi(X)$ and $M_u = \varphi^{-1}(u)$.

The following easy observation describes the Auslander-Reiten translation and the arrows of $\mathit{\Gamma}_{\mathscr{C}(Q)}$ in terms of the arcs in $\mathcal{B}_\infty$. Recall that $\mathrm{arc}(\mathcal{B}_\infty)$ is equipped with a translation τ as defined in Definition 3.1.

4.2. Lemma. *Let u, v be distinct arcs in $\mathcal{B}_\infty$, and let X be an object in $\mathscr{F}(Q)$.*

(1) *We have* $\tau_{\mathscr{C}} M_u = M_{\tau u}$, $\tau^-_{\mathscr{C}} M_u = M_{\tau^- u}$; *and* $\tau a_X = a_{\tau_{\mathscr{C}} X}$, $\tau^- a_X = a_{\tau^-_{\mathscr{C}} X}$.
(2) *If* $u = [\mathfrak{l}_i, \mathfrak{r}_j]$, *then there exists an arrow* $M_u \to M_v$ *in* $\Gamma_{\mathscr{C}(Q)}$ *if and only if* $v = [\mathfrak{l}_i, \mathfrak{r}_{j-1}]$ *or* $v = [\mathfrak{l}_{i-1}, \mathfrak{r}_j]$.
(3) *If* $u = [\mathfrak{l}_i, \mathfrak{l}_j]$ *with* $i \le j-2$, *then there exists an arrow* $M_u \to M_v$ *in* $\Gamma_{\mathscr{C}(Q)}$ *if and only if* $v = [\mathfrak{l}_i, \mathfrak{l}_{j-1}]$ *with* $i < j-2$ *or* $v = [\mathfrak{l}_{i-1}, \mathfrak{l}_j]$.
(4) *If* $u = [\mathfrak{r}_i, \mathfrak{r}_j]$ *with* $i \ge j+2$, *then there exists an arrow* $M_u \to M_v$ *in* $\Gamma_{\mathscr{C}(Q)}$ *if and only if* $v = [\mathfrak{r}_{i-1}, \mathfrak{r}_j]$ *with* $i > j+2$ *or* $v = [\mathfrak{r}_i, \mathfrak{r}_{j-1}]$.

The following result says that rigid pairs of indecomposable objects of $\mathscr{C}(Q)$ correspond to non-crossing pairs of arcs in $\mathcal{B}_\infty$.

4.3. THEOREM. *Let* u, v *be arcs in* $\mathcal{B}_\infty$. *If* M_u, M_v *are the corresponding objects in* $\mathscr{F}(Q)$, *then* (u, v) *is a crossing pair if and only if* $\mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_v[1]) \ne 0$.

Proof. By Corollary 2.10, we may assume that $u \ne v$. If one of u, v is an upper arc and the other one is a lower arc, then u, v do not cross. On the other hand, one of M_u, M_v lies in $\mathcal{R}_L$ and the other lies in $\mathcal{R}_R$. The result follows from Proposition 2.9(3) in this case.

Consider the case where u, v are connecting arcs. Then $M_u, M_v \in \mathcal{C}_Q$. There exists no loss of generality in assuming that M_u and $\tau_{\mathscr{C}} M_v = M_{\tau v}$ belong to $\mathcal{P}_Q$. Recall that $\mathcal{C}_Q$ is a standard component of $\Gamma_{D^b(\mathrm{rep}(Q))}$ of shape $\mathbb{Z}\mathbb{A}_\infty^\infty$; see [24, (2.3)] and [3, (7.9)]. Suppose first that (u, v) is crossing. We may assume that $u = [\mathfrak{l}_p, \mathfrak{r}_q]$ and $v = [\mathfrak{l}_i, \mathfrak{r}_j]$ with $i < p$ and $j < q$. By Lemma 4.2(1), $\mathcal{C}_Q$ contains a path

$$M_u = M_{[\mathfrak{l}_p, \mathfrak{r}_q]} \longrightarrow M_{[\mathfrak{l}_p, \mathfrak{r}_{q-1}]} \longrightarrow \cdots \longrightarrow M_{[\mathfrak{l}_p, \mathfrak{r}_{j+1}]} \longrightarrow M_{[\mathfrak{l}_{p-1}, \mathfrak{r}_{j+1}]}$$
$$\longrightarrow M_{[\mathfrak{l}_{p-2}, \mathfrak{r}_{j+1}]} \longrightarrow \cdots \longrightarrow M_{[\mathfrak{l}_{i+1}, \mathfrak{r}_{j+1}]} = M_{\tau v},$$

lying in the forward rectangle of M_u. Then, $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_u, M_{\tau v}) \ne 0$ by Proposition 1.1, and consequently, $\mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_{\tau v}) \ne 0$.

Suppose conversely that $\mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_{\tau v}) \ne 0$. Since $M_u, M_{\tau v}$ are assumed to be representations, by Lemma 2.6(1), either $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_u, M_{\tau v}) \ne 0$ or $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_{\tau v}, \tau_D^2 M_u) \ne 0$. Since $\mathcal{C}_Q$ is standard in $D^b(\mathrm{rep}(Q))$, we obtain a path $M_u \rightsquigarrow M_{\tau v}$ or $M_v \rightsquigarrow M_{\tau u}$, that is, a path $M_{[\mathfrak{l}_p, \mathfrak{r}_q]} \rightsquigarrow M_{[\mathfrak{l}_{i+1}, \mathfrak{r}_{j+1}]}$ or $M_{[\mathfrak{l}_i, \mathfrak{r}_j]} \rightsquigarrow M_{[\mathfrak{l}_{p+1}, \mathfrak{r}_{q+1}]}$ in $\mathcal{C}_Q$. By Lemma 4.2(1), $p \le i+1$ and $q \le j+1$ in the first case, and $i \le p+1$ and $j \le q+1$ in the second case. Thus, (u, v) is a crossing pair.

Consider now the case where v, u are upper arcs, say $u = [\mathfrak{l}_p, \mathfrak{l}_q]$ and $v = [\mathfrak{l}_i, \mathfrak{l}_j]$ with $p \le q-2$ and $i \le j-2$. Then $M_u, M_v \in \mathcal{R}_L$. Recall that $\mathcal{R}_L$ is a standard component of $\Gamma_{D^b(\mathrm{rep}(Q))}$ of shape $\mathbb{Z}\mathbb{A}_\infty$; see [24, (2.3)] and [3, (7.9)]. Assume that u crosses v, say $i < p < j < q$. By Lemma 4.2(3), $\mathcal{R}_L$ contains a path

$$M_u = M_{[\mathfrak{l}_p, \mathfrak{l}_q]} \longrightarrow M_{[\mathfrak{l}_p, \mathfrak{l}_{q-1}]} \longrightarrow \cdots \longrightarrow M_{[\mathfrak{l}_p, \mathfrak{l}_{j+2}]} \longrightarrow M_{[\mathfrak{l}_p, \mathfrak{l}_{j+1}]}$$
$$\longrightarrow M_{[\mathfrak{l}_{p-1}, \mathfrak{l}_{j+1}]} \longrightarrow M_{[\mathfrak{l}_{p-2}, \mathfrak{l}_{j+1}]} \longrightarrow \cdots \longrightarrow M_{[\mathfrak{l}_{i+1}, \mathfrak{l}_{j+1}]} = M_{\tau v},$$

lying in the forward rectangle of M_u. By Proposition 1.1, $\mathrm{Hom}_{\mathrm{rep}(Q)}(M_u, M_{\tau v}) \ne 0$, and consequently, $\mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_{\tau v}) \ne 0$.

Conversely, assume that $\mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_v[1]) = \mathrm{Hom}_{\mathscr{C}(Q)}(M_u, M_{\tau v}) \ne 0$. By Lemma 2.6(1), $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_u, M_{\tau v}) \ne 0$ or $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_{\tau v}, \tau_D^2 M_u) \ne 0$. Suppose that the first case occurs. By Proposition 1.1, $M_{\tau v}$ lies in the forward rectangle of M_u. Hence, $\mathcal{R}_L$ has an almost sectional path $M_u = M_{[\mathfrak{l}_p, \mathfrak{l}_q]} \rightsquigarrow M_{[\mathfrak{l}_{i+1}, \mathfrak{l}_{j+1}]}$, the composite of two paths $M_{[\mathfrak{l}_p, \mathfrak{l}_q]} \rightsquigarrow M_{[\mathfrak{l}_p, \mathfrak{l}_{j+1}]}$ and $M_{[\mathfrak{l}_p, \mathfrak{l}_{j+1}]} \rightsquigarrow M_{[\mathfrak{l}_{i+1}, \mathfrak{l}_{j+1}]}$. This gives rise to $i < p < j < q$. If the second case occurs, then $p < i < q < j$. Thus, (u, v) is a crossing pair. Similarly, we can treat the case where u, v are lower arcs.

Consider next the case where $u = [\mathfrak{l}_p, \mathfrak{l}_q]$ with $p \le q-2$ and $v = [\mathfrak{l}_i, \mathfrak{r}_j]$. By definition, we obtain $M_u = L^-_{p+1} \cap L^+_{q-1} \in \mathcal{R}_L$ and $M_v = L_i \cap R_j \in \mathcal{C}_Q$. Since $\tau_{\mathscr{C}}$ is an automorphism of $\mathscr{C}(Q)$, we may assume that $M_v \in \mathcal{P}_Q$. Since $M_u[1] = \tau_{\mathscr{C}} M_u \in \mathcal{R}_L$, by Corollary 2.7, $\mathrm{Hom}_{\mathscr{C}(Q)}(M_v, M_u[1]) \ne 0$ if and only if $\mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(M_v, \tau_{\mathscr{C}} M_u) \ne 0$. Since $\tau_{\mathscr{C}} M_u = M_{\tau u} = M_{[\mathfrak{l}_{p+1}, \mathfrak{l}_{q+1}]}$, by Proposition 4.1(2), the latter condition is equivalent to $M_{[\mathfrak{l}_{p+1}, \mathfrak{l}_{q+1}]} \in \mathcal{W}(\tau^i_{\mathscr{C}} M(q_0))$. Since $\tau^i_{\mathscr{C}} M(q_0) = \tau^{i+1}_{\mathscr{C}} S_L = L^+_{i+1} \cap L^-_{i+1}$ and $M_{[\mathfrak{l}_{p+1}, \mathfrak{l}_{q+1}]} = L^-_{p+2} \cap L^+_q$, we see that $M_{[\mathfrak{l}_{p+1}, \mathfrak{l}_{q+1}]} \in \mathcal{W}(\tau^i_{\mathscr{C}} M(q_0))$ if and only if $i+1 \ge p+2$ and $q \ge i+1$, that is, $p < i < q$. This last condition is evidently equivalent to u, v crossing. The case where u is a lower arc and v is a connecting arc can be treated in a similar manner. The proof of the theorem is completed.

The following statement is an alternative interpretation of Theorem 4.3.

4.4. Corollary. *Let X, Y be objects in $\mathscr{F}(Q)$. Then* $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y) \ne 0$ *if and only if $(a_Y, \tau a_X)$ is a crossing pair if and only if $(a_X, \tau^- a_Y)$ is crossing.*

Proof. The second equivalence is evident. Since $\mathscr{C}(Q)$ is 2-Calabi-Yau, we have

$$\mathrm{Hom}_{\mathscr{C}(Q)}(Y, \tau_{\mathscr{C}} X[1]) = \mathrm{Hom}_{\mathscr{C}(Q)}(Y, X[2]) \cong D\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y).$$

By definition, $Z = M_{a_Z}$ for every object $Z \in \mathscr{F}(Q)$. By Theorem 4.3, a_Y crosses $a_{\tau_{\mathscr{C}} X} = \tau a_X$ if and only if $\mathrm{Hom}_{\mathscr{C}(Q)}(Y, \tau_{\mathscr{C}} X[1]) \ne 0$, that is, $\mathrm{Hom}_{\mathscr{C}(Q)}(X, Y) \ne 0$. The proof of the corollary is completed.

Given a strictly additive subcategory $\mathcal{T}$ of $\mathscr{C}(Q)$, we shall write $\mathrm{arc}(\mathcal{T})$ for the set of arcs a_T with $T \in \mathcal{T} \cap \mathscr{F}(Q)$. As an immediate consequence of Theorem 4.3 and Lemma 2.11, we obtain the following result.

4.5. Theorem. *Let $\mathcal{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. Then $\mathcal{T}$ is weakly cluster-tilting if and only if* $\mathrm{arc}(\mathcal{T})$ *is a triangulation of $\mathcal{B}_\infty$.*

Our main objective is to determine the triangulations of $\mathcal{B}_\infty$ which correspond to cluster-tilting subcategories of $\mathscr{C}(Q)$. For this purpose, the following technical result is needed.

4.6. Lemma. *Let $f : M \to N$ and $g : N \to L$ be non-zero morphisms between indecomposable objects in $\mathscr{C}(Q)$. If* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, N[1]) = 0$, *then* $\mathrm{Hom}_{\mathscr{C}(Q)}(M, L)$ *is generated by gf over k.*

Proof. Suppose that $\{M, N\}$ is a rigid pair and that $\mathrm{Hom}_{\mathscr{C}(Q)}(M, L) \ne 0$. By Proposition 2.12, it suffices to show that $gf \ne 0$. Since $\tau_{\mathscr{C}}$ ia an auto-equivalence, we may assume that $\tau^i_{\mathscr{C}} M, \tau^i_{\mathscr{C}} N, \tau^i_{\mathscr{C}} L \in \mathit{\Gamma}_{\mathrm{rep}(Q)}$, for $-1 \le i \le 1$. Let $\mathit{\Delta}$ be a connected finite full subquiver of Q which supports all these representations and is closed under taking successors. Then, $\tau^i_{\mathit{\Delta}}(X) = \tau^i_Q(X)$ for $-1 \le i \le 1$. Since every projective representation in $\mathrm{rep}(\mathit{\Delta})$ is projective in $\mathrm{rep}(Q)$, moreover, $D^b(\mathrm{rep}(\mathit{\Delta}))$ is a full triangulated subcategory of $D^b(\mathrm{rep}(Q))$; see [2, (1.11)]. Let $F_{\mathit{\Delta}} = \tau^- \circ [1]$, where τ is the Auslander-Reiten translation of $D^b(\mathrm{rep}(\mathit{\Delta}))$. For $X, Y \in \{M, N, L\}$, we have $FY = F_{\mathit{\Delta}} Y$, and as seen in the proof of Lemma 2.6,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}(Q)}(X,Y) &= \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,Y)\oplus \mathrm{Hom}_{D^b(\mathrm{rep}(Q))}(X,FY)\\ &\cong \mathrm{Hom}_{D^b(\mathrm{rep}(\varDelta))}(X,Y)\oplus \mathrm{Hom}_{D^b(\mathrm{rep}(\varDelta))}(X,F_{\varDelta}Y).\end{aligned}$$

Since $\varDelta$ is of type $\mathbb{A}_n$ for some $n>0$, it is well known that $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$ is standard of shape $\mathbb{Z}\mathbb{A}_n$; see [12, (5.5)]. By the assumption, there exists an object $N_1 \in \{N, F_{\varDelta}N\}$ and objects $L_1, L_2 \in \{L, F_{\varDelta}L\}$ such that N_1, L_1 lies in the forward rectangle of M, and L_2 lies in the forward rectangle of N_1, in $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$.

Observing that $\tau N_1 \cong \tau_{\mathscr{C}} N$ in $\mathscr{C}(Q)$, by the rigidity of (M,N) in $\mathscr{C}(Q)$, we obtain $\mathrm{Hom}_{D^b(\mathrm{rep}(\varDelta))}(M,\tau N_1)=0$. Thus, $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$ contains a sectional path $M \rightsquigarrow N_1$, which is contained in a maximal sectional path $M \rightsquigarrow N_1 \rightsquigarrow S$, where S has only one immediate predecessor in $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$. Then, M, N_1, L_1, L_2 all lie in the wing $\mathcal{W}(S)$ with wing vertex S; see, for definition, [26, (3.3)]. It is easy to see that every wing in $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$ meets each $F_{\varDelta}$-orbit exactly once. In particular, $L_2 = L_1$ lies in the forward rectangle of M. In this case, the composite of any path from M to N_1 and any path from N_1 to L_2 in $\varGamma_{D^b(\mathrm{rep}(\varDelta))}$ contains no monomial mesh relation. In particular, $gf \neq 0$. The proof of the lemma is completed.

We are ready to obtain the main result of this section, which characterizes the cluster-tilting subcategories of $\mathscr{C}(Q)$ in terms of the triangulations of $\mathcal{B}_\infty$.

4.7. Theorem. *Let Q be a quiver of type $\mathbb{A}_\infty^\infty$ with no infinite path, and let $\mathcal{T}$ be a strictly additive subcategory of $\mathscr{C}(Q)$. The following statements are equivalent.*

(1) *The subcategory $\mathcal{T}$ is cluster-tilting.*
(2) *The set* $\mathrm{arc}(\mathcal{T})$ *is a compact triangulation of $\mathcal{B}_\infty$.*
(3) *The set* $\mathrm{arc}(\mathcal{T})$ *is a triangulation containing infinitely many connecting arcs, and every marked point in $\mathcal{B}_\infty$ is* $\mathrm{arc}(\mathcal{T})$*-bounded or an* $\mathrm{arc}(\mathcal{T})$*-fountain base.*

In this case, moreover, $\mathrm{arc}(\mathcal{T})$ *has at most two fountains, and if it has two, then one of them is a left fountain and the other one is a right fountain.*

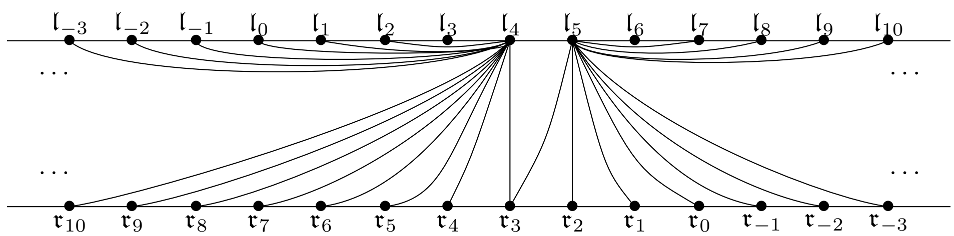
Proof. In view of Theorem 3.18 and Lemmas 3.14 and 3.17, it suffices to show the equivalence of Statements (1) and (2). By Theorem 4.5, it amounts to show that $\mathcal{T}$ is functorially finite in $\mathscr{C}(Q)$ if and only if $\mathrm{arc}(\mathcal{T})$ is compact in case $\mathcal{T}$ is weakly cluster-tilting. Let this be the case.

Assume first that $\mathrm{arc}(\mathcal{T})$ is compact. Let X be an indecomposable object of $\mathscr{C}(Q)$. Denote by $\varOmega$ the set of arcs of $\mathrm{arc}(\mathcal{T})$ crossing $\tau^- a_X$, which is compact by the assumption. Let $\varSigma$ be a finite subset of $\varOmega$ satisfying the condition stated in Definition 3.9. For each $v \in \varSigma$, since $a_{M_v} = v$ crosses $\tau^- a_X$, we may find a nonzero morphism $f_v : M_v \to X$ in $\mathscr{C}(Q)$ by Corollary 4.4. We claim that $f = \oplus_{v\in\varSigma} f_v : \oplus_{v\in\varSigma} M_v \to X$ is a right $\mathcal{T}$-approximation for X. Indeed, let $T \in \mathcal{T}$ be indecomposable with $\mathrm{Hom}_{\mathscr{C}(Q)}(T,X)\neq 0$. By Corollary 4.4, a_T crosses $\tau^- a_X$, that is, $a_T \in \varOmega$. Then, there exists $w \in \varSigma$ such that a_T crosses $\tau^- w = \tau^- a_{M_w}$. By Corollary 4.4, we can find a non-zero morphism $g_w : T \to M_w$ in $\mathscr{C}(Q)$. Consider the chosen nonzero morphism $f_w : M_w \to X$. By Lemma 4.6, every morphism $g : T \to X$ is a multiple of $f_w g_w$. In particular, g factors through f. This establishes our claim. Therefore, $\mathcal{T}$ is contravariantly finite in $\mathscr{C}(Q)$. Using the dual of Lemma 4.6 and the compactness of the set of arcs of $\mathrm{arc}(\mathcal{T})$ crossing τa_X, we may show that $\mathcal{T}$ is covariantly finite in $\mathscr{C}(Q)$.

Suppose conversely that $\mathcal{T}$ is functorially finite in $\mathscr{C}(Q)$. Let $u \in \mathrm{arc}(\mathcal{B}_\infty)$. By the assumption, $M_{\tau u}$ admits a minimal right $\mathcal{T}$-approximation $f : T \to M_{\tau u}$.

We may write $T = \oplus_{w\in\Sigma^-} M_w$, where Σ^- is a finite subset of $\mathrm{arc}(\mathcal{T})$. For each $w \in \Sigma^-$, restricting f to M_w yields a non-zero morphism $f_w : M_w \to M_{\tau u}$. By Corollary 4.4, $a_{M_w} = w$ crosses $\tau^- a_{M_{\tau u}} = u$. This shows that $\Sigma^- \subseteq \mathrm{arc}(\mathcal{T})_u$. Now, for each $v \in \mathrm{arc}(\mathcal{T})_u$, since $a_{M_v} = v$ crosses $\tau^- a_{M_{\tau u}} = u$, we deduce from Corollary 4.4 that there exists a nonzero morphism $g : M_v \to M_{\tau u}$. Then g factors through $f : \oplus_{w\in\Sigma^-} M_w \to M_{\tau u}$. In particular, $\mathrm{Hom}_{\mathscr{C}(Q)}(M_v, M_{v_1}) \neq 0$ for some $v_1 \in \Sigma^-$. By Corollary 4.4, v crosses $\tau^- v_1$. Similarly, considering a minimal left $\mathcal{T}$-approximation for $M_{\tau^- u}$, we obtain a finite subset Σ^+ of $\mathrm{arc}(\mathcal{T})_u$ such that each arc v of $\mathrm{arc}(\mathcal{T})_u$ crosses some arc of Σ^+. Then $\Sigma = \Sigma^- \cup \Sigma^+$ is a finite subset of $\mathrm{arc}(\mathcal{T})_u$ satisfying the condition stated in Definition 3.9. Thus, $\mathrm{arc}(\mathcal{T})_u$ is compact. This shows that $\mathrm{arc}(\mathcal{T})$ is compact. The proof of the theorem is completed.

Example. The following picture shows a compact triangulation of $\mathcal{B}_\infty$ with two fountains, which corresponds to a cluster-tilting subcategory of $\mathscr{C}(Q)$.



We would like to conclude the paper with a final remark. Let $\mathcal{T}$ be a cluster-tilting subcategory of $\mathscr{C}(Q)$ with an indecomposable object M. We know that there exists a unique (up to isomorphism) indecomposable object M^* in $\mathscr{C}(Q)$ but not in $\mathcal{T}$ such that the additive subcategory generated by $\mathcal{T}_M$ and M^* is cluster-tilting. On the geometric side, $\mathrm{arc}(\mathcal{T})$ is a compact triangulation of $\mathcal{B}_\infty$ and a_M is a side of exactly two triangles, that is, a_M is a diagonal of a quadrilateral formed by some arcs in $\mathrm{arc}(\mathcal{T})$ and some edges in $\mathcal{B}_\infty$. It is easy to see that the other diagonal u of the quadrilateral together with the arcs in $\mathrm{arc}(\mathcal{T})\backslash\{a_M\}$ form a triangulation of $\mathcal{B}_\infty$ satisfying the condition stated in Theorem 4.7(3). By the uniqueness, we obtain $a_{M^*} = u$. In other words, mutation corresponds to arc flipping.

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Shiping Liu, Département de mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada

E-mail address: shiping.liu@usherbrooke.ca

Charles Paquette, Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

E-mail address: charles.paquette@usherbrooke.ca