

REPRESENTATION THEORY OF STRONGLY LOCALLY FINITE QUIVERS

RAYMUNDO BAUTISTA, SHIPING LIU, AND CHARLES PAQUETTE

ABSTRACT. This paper deals with the representation theory of strongly locally finite quivers. We first study some properties of the finitely presented or co-presented representations, and then construct in the category of locally finite dimensional representations some almost split sequences which start with a finitely co-presented representation and end with a finitely presented representation. Furthermore, we obtain a general description of the shapes of the Auslander-Reiten components of the category of finitely presented representations and prove that the number of regular Auslander-Reiten components is infinite if and only if the quiver is not of finite or infinite Dynkin type. In the infinite Dynkin case, we shall give a complete list of the indecomposable representations and an explicit description of the Auslander-Reiten components. Finally, we apply these results to study the Auslander-Reiten theory in the derived category of bounded complexes of finitely presented representations.

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INTRODUCTION

As the best understood and the most stimulating part of the representation theory of finite dimensional algebras, the theory of representations of finite quivers has been extensively studied over the last four decades; see, for example, [11, 18, 19, 26, 28, 40, 41]. On the other hand, the Auslander-Reiten theory of irreducible morphisms and almost split sequences provides an indispensable powerful tool for the representation theory and it appears in many other areas such as algebraic geometry and algebraic topology; see [6, 3, 25]. The impact of these two theories to other branches of mathematics is best illustrated by their recent interaction with the theory of cluster algebras via the cluster category; see, for example, [12, 17]. Now, new developments require the study of representations of infinite quivers. For instance, in order to classify the noetherian Ext-finite hereditary abelian categories with Serre duality, Reiten and Van den Bergh investigated the category of finitely presented representations of a locally finite quiver without left infinite paths; see [39]. In particular, they showed that this category has right almost split sequences and obtained some partial description of its Auslander-Reiten components. Later, by considering representations of ray quivers which are infinite in general, Ringel provided an alternative construction of the noetherian Ext-finite hereditary abelian categories which have Serre duality and non-zero projective objects; see [43]. More recently, Holm and Jørgensen studied a cluster category of infinite Dynkin type, which can be constructed from the category of finite dimensional representations of a quiver of type \mathbb{A}_∞ and whose Auslander-Reiten quiver is of shape $\mathbb{Z}\mathbb{A}_\infty$; see [24]. Furthermore, the bounded derived category of a finite dimensional algebra with radical squared zero is determined by the category of finitely co-presented representations of a covering of the ordinary quiver of the algebra which is usually infinite; see [10], and homogeneous vector bundles over certain algebraic variety

are characterized by the finite dimensional representations of some infinite quiver; see [23]. Finally, representations of infinite quivers have been playing an important role in the representation theory of coalgebras; see [14, 15, 37]. The latter in turn is useful in the study of the Jacobian algebra of a quiver with potentials, which is endowed with a natural structure of a topological algebra; see [27, 44].

All these motivate us to study representations of quivers in the most general setting. Indeed, we shall work with an arbitrary base field k and a quiver Q which is assumed only to be locally finite such that the number of paths between any given pair of vertices is finite. The main objective of this paper is to present a complete picture of the Auslander-Reiten theory in the category of finitely presented k -representations of Q and in its bounded derived category. Our results yield many interesting examples of Ext-finite, but not necessarily noetherian, hereditary abelian categories which have (left, right) almost split sequences. We outline the content of the paper section by section as follows.

In Section 1, we study almost finitely presented and almost finitely co-presented representations; see (1.5), which are slightly more general than finitely presented and finitely co-presented representations, respectively. We shall give some combinatorial characterizations of these representations; see (1.12). These will be useful for us to relate almost finitely presented or co-presented representations of Q to representations of its finite subquivers; see (1.13).

In Section 2, we study the Auslander-Reiten theory in the category $\text{rep}(Q)$ of locally finite dimensional k -representations of Q . We first construct almost split sequences which start with a finitely co-presented representation and end with a finitely presented one; see (2.8), and then study some important properties of the Auslander-Reiten orbits in $\text{rep}(Q)$; see (2.14). Finally, we develop some tools to relate almost split sequences and irreducible morphisms in $\text{rep}(Q)$ to those of locally finite dimensional representations of subquivers of Q ; see (2.16) and (2.17).

In Section 3, we start to concentrate on the study of the Auslander-Reiten theory in the category $\text{rep}^+(Q)$ of finitely presented k -representations of Q . We shall show that irreducible morphisms between indecomposable representations in $\text{rep}^+(Q)$ are irreducible in $\text{rep}(Q)$; see (3.5), and almost split sequences in $\text{rep}^+(Q)$ have finite dimensional starting term and remain almost split in $\text{rep}(Q)$; see (3.6). This will enable us to find necessary and sufficient conditions so that $\text{rep}^+(Q)$ has (left, right) almost split sequences; see (3.7) and (3.8).

In Section 4, we shall give a general description of the Auslander-Reiten quiver $\Gamma_{\text{rep}^+(Q)}$ of $\text{rep}^+(Q)$, which is defined since $\text{rep}^+(Q)$ is Krull-Schmidt; see [33, (2.1)]. In case Q is connected, $\Gamma_{\text{rep}^+(Q)}$ has a unique preprojective component which is a predecessor-closed subquiver of $\mathbb{N}Q^{\text{op}}$; see (4.6). The preinjective component correspond to the connected components of the subquiver of Q generated by the vertices which are not ending point of any left infinite paths, and they are finite or infinite successor-closed subquiver of $\mathbb{N}^-Q^{\text{op}}$; see (4.7). In particular, the number of preinjective components varies from zero to infinity. The other connected components of $\Gamma_{\text{rep}^+(Q)}$, called *regular components*, are of shape \mathbb{N}^-A_∞ , $\mathbb{N}A_\infty$, $\mathbb{Z}A_\infty$, or finite wings; see (4.14). As a consequence, $\Gamma_{\text{rep}^+(Q)}$ is always symmetrically valued. At the end of this section, some conditions on Q will be given so that at most one type of regular components will appear; see (4.16) and (4.17).

In Section 5, we study the representation theory of infinite Dynkin types \mathbb{A}_∞ , \mathbb{A}_∞^∞ , and \mathbb{D}_∞ . As in the finite Dynkin case, we shall obtain a complete list of the indecomposable representations in $\text{rep}^+(Q)$; see (5.9) and (5.19). In case Q is of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ , we shall describe the irreducible morphisms between indecomposable representations and determine the almost split sequences with an indecomposable middle term; see (5.10) and (5.12). This allows us to give an explicit description of the connected components of $\Gamma_{\text{rep}^+(Q)}$ for each of the three infinite Dynkin types; see (5.16), (5.17), and (5.22). In summary, $\Gamma_{\text{rep}^+(Q)}$ has at most four connected components of which at most two are regular.

In Section 6, we prove that $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components in case Q is not of finite or infinite Dynkin type; see (6.4). Moreover, we will show that each of the four types of regular components could appear infinitely many times in certain particular situation; see (6.6).

In Section 7, we study the Auslander-Reiten theory in the bounded derived category $D^b(\text{rep}^+(Q))$ of $\text{rep}^+(Q)$. Indeed, for an arbitrary hereditary abelian category \mathcal{H} , we shall describe the so-called *connecting almost split triangles* in $D^b(\mathcal{H})$ and prove that all other almost split triangles are induced from the almost split sequences in \mathcal{H} ; see (7.2). Combining this with previously obtained results, we get a general description of the Auslander-Reiten quiver of $D^b(\text{rep}^+(Q))$; see (7.7), (7.9) and (7.10). We conclude the paper with some necessary and sufficient conditions on Q so that $D^b(\text{rep}^+(Q))$ has (left, right) almost split triangles; see (7.11).

1. ALMOST FINITELY PRESENTED REPRESENTATIONS

The objective of this section is to investigate two classes of representations, called *almost finitely presented* and *almost finitely co-presented*, of a quiver over a field. This will yield several hereditary abelian categories, some of them are Krull-Schmidt in which we shall be able to study the Auslander-Reiten theory. Since a direct sum of injective modules is not necessarily injective, we shall concentrate on almost finitely co-presented representations while the corresponding results for almost finitely presented representations will follow dually.

We start by introducing some combinatorial terminology and notation. Throughout this paper, $Q = (Q_0, Q_1)$ stands for a quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. Let $\alpha : x \rightarrow y$ be an arrow in Q . We call x the *starting point* and y the *ending point* of α , and write $s(\alpha) = x$ and $e(\alpha) = y$. One introduces a formal inverse α^{-1} with $s(\alpha^{-1}) = y$ and $e(\alpha^{-1}) = x$. An *edge* in Q is an arrow or the inverse of an arrow. To each vertex x in Q , one associates a *trivial path*, also called *trivial walk*, ε_x with $s(\varepsilon_x) = e(\varepsilon_x) = x$. A *non-trivial walk* w in Q is a finite or infinite product

$$\cdots c_{i+1}c_i \cdots,$$

where the c_i are edges such that $e(c_i) = s(c_{i+1})$ for all i , whose *inverse* w^{-1} is defined to be the product

$$\cdots c_i^{-1}c_{i+1}^{-1} \cdots.$$

Such a walk w is called *reduced* if $c_{i+1} \neq c_i^{-1}$ for every i , and *acyclic* if it passes through any given vertex in Q at most once. Clearly an acyclic walk is reduced. If w is a finite or infinite product

$$\cdots c_i \cdots c_2c_1,$$

then we call c_1 the *initial edge* of w , we define $s(w) = s(c_1)$, the *starting point* of w , and we write $w \cdot \varepsilon_{s(w)} = w$. Dually, if w is a finite or infinite product

$$c_1 c_2 \cdots c_i \cdots,$$

then we call c_1 the *terminal edge* of w , we define $e(w) = e(c_1)$, the *ending point* of w , and we write $\varepsilon_{s(w)} \cdot w = w$. An infinite walk w is called *right infinite* if $s(w)$ is defined, *left infinite* if $e(w)$ is defined, and *double infinite* if neither $s(w)$ nor $e(w)$ is defined. Clearly, w is finite if and only if both $s(w)$ and $e(w)$ are defined, and in this case, we call w a walk from $s(w)$ to $e(w)$.

A non-trivial walk in Q is called a *path* if all of its edges are arrows. A *middle point* of a path is a vertex which appears in the path but is neither the starting point nor the ending point. A path is called *maximal* if it is not a proper subpath of any other path in Q . Let x, y be vertices in Q . If Q contains a path p from x to y , then we say that x is a *predecessor* of y which is *trivial* or *immediate* if p is a trivial path or an arrow, respectively; while y is a *successor* of x which is *trivial* or *immediate* if p is a trivial path or an arrow, respectively.

For $x \in Q_0$, we denote by x^+ and x^- the set of arrows starting in x and the set of arrows ending in x , respectively. We say that x is a *sink vertex* or a *source vertex* if $x^+ = \emptyset$ or $x^- = \emptyset$, respectively. Moreover, one says that Q is *locally finite* if x^+ and x^- are finite for any $x \in Q_0$, and in this case, one defines the *weight* of x to be the sum of the cardinalities of x^- and x^+ . For $x, y \in Q_0$, let $Q(x, y)$ stand for the set of paths in Q from x to y . One says that Q is *interval-finite* if $Q(x, y)$ is finite for any $x, y \in Q_0$. For short, we say that Q is *strongly locally finite* if it is locally finite and interval-finite. Note that Q contains no oriented cycle in case it is interval-finite.

Let Σ be a subquiver of Q . We shall say that Σ is *full* if, for any vertices x, y in Σ , all the arrows in Q from x to y lie in Σ ; *convex* if, for any vertices x, y in Σ , all the paths in Q from x to y lie in Σ ; *predecessor-closed* if every path in Q ending in some vertex in Σ lies entirely in Σ ; and *successor-closed* if every path in Q starting in some vertex in Σ lies entirely in Σ .

From now on, k denotes an arbitrary field and Q is a strongly locally finite quiver. We shall compose morphisms in any category from the right to the left, and all tensor products are over k . A *representation* M of Q over k , or simply a *k-representation*, consists of a family of k -spaces $M(x)$ with $x \in Q_0$, and a family of k -maps $M(\alpha) : M(x) \rightarrow M(y)$ with $\alpha : x \rightarrow y$ in Q_1 . For each path ρ in Q , we write $M(\rho) = \mathbf{1}_{M(x)}$ if $\rho = \varepsilon_x$ and $M(\rho) = M(\alpha_r) \cdots M(\alpha_1)$ if $\rho = \alpha_r \cdots \alpha_1$ with $\alpha_1, \dots, \alpha_r \in Q_1$. Recall that a morphism $f : M \rightarrow N$ of k -representations of Q consists of a family of k -maps $f(x) : M(x) \rightarrow N(x)$ with $x \in Q_0$ such that $f(y)M(\alpha) = N(\alpha)f(x)$, for every arrow $\alpha : x \rightarrow y$. Let M be a k -representation of Q . The *socle* of M , written as $\text{soc}M$, is the sub-representation of M so that $(\text{soc}M)(x)$, for any $x \in Q_0$, is the intersection of the kernels of the maps $M(\alpha)$ with $\alpha \in x^+$; the *radical* of M , written as $\text{rad}M$, is the sub-representation of M such that $(\text{rad}M)(x)$, for any $x \in Q_0$, is the sum of the images of the maps $M(\beta)$ with $\beta \in x^-$; and the *top* of M , written as $\text{top}M$, is the quotient $M/\text{rad}M$. We shall say that $\text{top}M$ is *essential* over M if $\text{rad}M$ is superfluous in M . Furthermore, the *support* of a representation M , written as $\text{supp}M$, is the full subquiver of Q generated by the vertices x for which $M(x) \neq 0$. One says that M is *sincere* if $\text{supp}M = Q$, *finitely*

supported if $\text{supp}M$ is finite, and supported by a subquiver Σ of Q if $\text{supp}M \subseteq \Sigma$. Finally, M is called *locally finite dimensional* if $M(x)$ is of finite k -dimension for all $x \in Q_0$; and *finite dimensional* if $\sum_{x \in Q_0} \dim M(x)$ is finite. We shall denote by $\text{Rep}(Q)$ the abelian category of all k -representations of Q , which is known to be hereditary, that is, $\text{Ext}^2(-, -)$ vanishes; see [19, (8.2)]. Moreover, $\text{Rep}^b(Q)$, $\text{rep}(Q)$ and $\text{rep}^b(Q)$ will stand for the full subcategories of $\text{Rep}(Q)$ generated by the finitely supported representations, by the locally finite dimensional representations, and by the finite dimensional representations, respectively.

The following easy observation will be used in our later investigation.

1.1. LEMMA. *Let M be an object in $\text{Rep}(Q)$. If $\text{supp}M$ has no right infinite path, then $\text{soc}M$ is essential in M . If $\text{supp}M$ has no left infinite path, then $\text{top}M$ is essential over M .*

Proof. We shall prove only the first part. Let N be a non-zero sub-representation of M . Let x be a vertex in $\text{supp}M$ such that $N(x)$ has a non-zero element v . If $\text{supp}M$ has no right infinite path, then $\text{supp}M$ has a maximal path ρ starting with x such that $N(\rho)(v) \neq 0$. Note that $N(\rho)(v)$ lies in $N \cap \text{soc}M$. The proof of the lemma is completed.

We shall introduce more notation which will be used throughout the paper. Let $a \in Q_0$. The *simple representation* S_a at a is defined by $S_a(a) = k\varepsilon_a$ and $S_a(x) = 0$ for all vertices $x \neq a$. Moreover, let P_a be the k -representation such that $P_a(x)$, for any $x \in Q_0$, is the k -space spanned by $Q(a, x)$; and $P_a(\alpha) : P_a(x) \rightarrow P_a(y)$, for $\alpha : x \rightarrow y \in Q_1$, is the k -map sending every path ρ to $\alpha\rho$. Finally, I_a is the k -representation such that $I_a(x)$, for $x \in Q_0$, is the k -space spanned by $Q(x, a)$; and $I_a(\alpha) : I_a(x) \rightarrow I_a(y)$, for $\alpha : x \rightarrow y \in Q_1$, is the k -map sending $\rho\alpha$ to ρ and vanishing on the paths which do not factor through α . Since Q is interval-finite, the representations P_a, I_a are locally finite dimensional.

1.2. LEMMA. *Let $I = I_a \otimes V$ with $a \in Q_0$ and V a non-zero k -space. If $x \in Q_0$ and $\alpha_1, \dots, \alpha_s$, $s \geq 1$, are the arrows in $\text{supp}I$ starting in x , then $I(x) = W_1 \oplus \dots \oplus W_s$ such that $I(\alpha_i)(W_i) \neq 0$ and $I(\alpha_i)(W_j) = 0$, for $1 \leq i, j \leq s$ with $i \neq j$. As a consequence, I has an essential socle $S_a \otimes V$.*

Proof. Let $x \in Q_0$ and $\alpha_1, \dots, \alpha_s$ be the arrows in $\text{supp}I$ starting with x . Then $I_a(x) = N_1 \oplus \dots \oplus N_s$, where N_i is spanned by the paths $x \rightsquigarrow a$ factoring through α_i . By definition, $I_a(\alpha_i)(N_i) \neq 0$ and $I_a(\alpha_i)(N_j) = 0$, for $1 \leq i, j \leq s$ with $i \neq j$. Since $I(x) = (N_1 \otimes V) \oplus \dots \oplus (N_s \otimes V)$ and $I(\alpha_i) = I_a(\alpha_i) \otimes \mathbf{1}_V$, the first part of the lemma follows. Since Q has no oriented cycle, $I_a(a) = k\varepsilon_a$, and for any vertex x in $\text{supp}I$, we see that $\text{supp}I$ has no arrow starting with x if and only if $x = a$. Thus $\text{soc}I = S_a \otimes V$. Furthermore, since Q is interval-finite, $\text{supp}I$ has no right infinite path. By Lemma 1.1, $\text{soc}I$ is essential in I . The proof of the lemma is completed.

1.3. PROPOSITION. *Let M be an object in $\text{Rep}(Q)$ and V be a k -space. For each $a \in Q_0$, there exist k -linear isomorphisms which are natural in M as follows:*

$$\phi_M : \text{Hom}_{\text{Rep}(Q)}(M, I_a \otimes V) \longrightarrow \text{Hom}_k(M(a), V),$$

$$\psi_M : \text{Hom}_{\text{Rep}(Q)}(P_a \otimes V, M) \longrightarrow \text{Hom}_k(V, M(a)).$$

Proof. We shall prove only the first part of the statement. Fix $a \in Q_0$. For each $x \in Q_0$, we have $(I_a \otimes V)(x) = \bigoplus_{\rho \in Q(x,a)} (k\rho \otimes V)$. Since Q has no oriented cycle, $I_a(a) = k\varepsilon_a$. Let $e_v : I_a(a) \otimes V \rightarrow V$ be the k -isomorphism such that $e_v(\varepsilon_a \otimes v) = v$, for all $v \in V$. Define

$$\phi_M : \text{Hom}_{\text{Rep}(Q)}(M, I_a \otimes V) \rightarrow \text{Hom}_k(M(a), V) : f \mapsto e_v f(a),$$

which is clearly k -linear and natural in M . If $f : M \rightarrow I_a \otimes V$ is a morphism such that $\phi_M(f) = 0$, then $f(a) = 0$. Hence $\text{soc}(I_a \otimes V) \cap \text{Im}(f) = 0$. Since $\text{soc}(I_a \otimes V)$ is essential in $I_a \otimes V$, we get $f = 0$. That is, ϕ_M is a monomorphism. For proving the surjectivity, let $g : M(a) \rightarrow V$ be a k -map. For each $x \in Q_0$, since $Q(x, a)$ is finite, we have a k -linear map

$$f(x) : M(x) \rightarrow I_a(x) \otimes V : v \mapsto \sum_{\rho \in Q(x,a)} \rho \otimes g(M(\rho)(v)).$$

Let $\alpha : x \rightarrow y$ be an arrow in Q . We claim that $(I_a(\alpha) \otimes \mathbf{1}_V)f(x) = f(y)M(\alpha)$. Indeed, if $Q(x, a) = \emptyset$, then $Q(y, a) = \emptyset$. Hence $f(x) = 0$ and $f(y) = 0$. Assume that $Q(x, a) \neq \emptyset$. Fix $v \in M(x)$. Then $f(x)(v) = \sum_{\rho \in Q(x,a)} \rho \otimes g(M(\rho)(v))$. If $\rho = \sigma\alpha$ with $\sigma \in Q(y, a)$, then $I_a(\alpha)(\rho) = \sigma$, and otherwise, $I_a(\alpha)(\rho) = 0$. Thus,

$$(I_a(\alpha) \otimes \mathbf{1}_V)(f(x)(v)) = \sum_{\sigma \in Q(y,a)} \sigma \otimes g(M(\sigma)M(\alpha)(v)) = f(y)(M(\alpha)(v)).$$

This establishes our claim. As a consequence, the $f(x)$ with $x \in Q_0$ form a morphism $f : M \rightarrow I_a \otimes V$ in $\text{Rep}(Q)$. By definition, $\phi_M(f) = e_v f(a) = g$. The proof of the proposition is completed.

It follows from the preceding result that $P_a \otimes V$ is projective and $I_a \otimes V$ is injective in $\text{Rep}(Q)$. Moreover, P_a and I_a are indecomposable. Let $\text{Inj}(Q)$ be the full additive subcategory of $\text{Rep}(Q)$ generated by the objects isomorphic to $I_a \otimes V_a$ with $a \in Q_0$ and V_a some k -space, and let $\text{Proj}(Q)$ be the one generated by the objects isomorphic to $P_a \otimes U_a$ with $a \in Q_0$ and U_a some k -space. Moreover, we denote by $\text{inj}(Q)$ and $\text{proj}(Q)$ the full additive subcategories of $\text{Inj}(Q)$ and $\text{Proj}(Q)$, respectively, generated by the locally finite dimensional representations.

1.4. COROLLARY. *If M is a representation in $\text{Rep}(Q)$, then*

- (1) $M \in \text{Inj}(Q)$ if and only if M is injective in $\text{Rep}(Q)$ with $\text{soc}M$ being finitely supported and essential in M ;
- (2) $M \in \text{Proj}(Q)$ if and only if M is projective in $\text{Rep}(Q)$ with $\text{top}M$ being finitely supported and essential over M .

Proof. We shall prove only Statement (1). The necessity follows from Lemma 1.2 and Proposition 1.3. For the sufficiency, let M be a non-zero injective object in $\text{Rep}(Q)$ such that $\text{soc}M$ is finitely supported and essential in M . Then $\text{soc}M = (S_{a_1} \otimes V_1) \oplus \cdots \oplus (S_{a_s} \otimes V_s)$, where the a_i are vertices in Q and the V_i are non-zero k -spaces. Set $I = (I_{a_1} \otimes V_1) \oplus \cdots \oplus (I_{a_s} \otimes V_s)$. Then $\text{soc}M = \text{soc}I$. Observing that I is injective by Proposition 1.3, we have a morphism $f : M \rightarrow I$ which acts identically on $\text{soc}M$. Since $\text{soc}M$ is essential, f is a monomorphism. Since M is injective, f is a section. Since $\text{soc}I$ is essential, f is an isomorphism. Thus $M \in \text{Inj}(Q)$. The proof of the corollary is completed.

1.5. DEFINITION. Let M be an object in $\text{Rep}(Q)$. We say that M is *almost finitely co-presented* if it admits an injective co-resolution

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

with $I_0, I_1 \in \text{Inj}(Q)$, and *finitely co-presented* if, in addition, $I_0, I_1 \in \text{inj}(Q)$. Dually, M is called *almost finitely presented* if it admits a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_1, P_0 \in \text{Proj}(Q)$, and *finitely presented* if, in addition, $P_1, P_0 \in \text{proj}(Q)$.

We shall say that a quiver is *top-finite* if each vertex is a successor of finitely many pre-fixed vertices; and *socle-finite* if each vertex is a predecessor of finitely many pre-fixed vertices.

1.6. LEMMA. *Let M be a representation in $\text{Rep}(Q)$.*

- (1) *If M is almost finitely co-presented, then $\text{soc } M$ is finitely supported and essential in M , and $\text{supp } M$ is socle-finite with no right infinite path.*
- (2) *If M is almost finitely presented, then $\text{top } M$ is finitely supported and essential over M , and $\text{supp } M$ is top-finite with no left infinite path.*

Proof. We shall prove only Statement (1). Assume that M is non-zero and almost finitely co-presented. Suppose that $M \rightarrow I$ is the injective envelope of M , where $I = (I_{a_1} \otimes U_1) \oplus \cdots \oplus (I_{a_r} \otimes U_r)$ with $a_1, \dots, a_r \in Q_0$ and U_1, \dots, U_r some non-zero k -spaces. Then $\text{soc } M$ is essential in M and supported by the vertices a_1, \dots, a_r . Let x be a vertex in $\text{supp } M$. Choose some non-zero element $v \in M(x)$ and let L be the sub-representation of M generated by v . Since $L \cap \text{soc } M \neq 0$, $\text{supp } L$ contains some a_i with $1 \leq i \leq r$. Therefore, $\text{supp } L$ has a path from x to a_i . That is, x is a predecessor of a_i in $\text{supp } M$. Therefore, $\text{supp } M$ is socle-finite. Since Q is interval-finite, we see that $\text{supp } M$ has no right infinite path. The proof of the lemma is completed.

The following result states some useful combinatorial properties of the finitely presented or finitely co-presented but infinite dimensional representations.

1.7. COROLLARY. *Let M be an infinite dimensional representation in $\text{rep}(Q)$.*

- (1) *If M is finitely presented, then $\text{supp } M$ contains a right infinite path.*
- (2) *If M is finitely co-presented, then $\text{supp } M$ contains a left infinite path.*

Proof. We shall prove only Statement (1). Assume that M is finitely presented. By Lemma 1.6, $\text{supp } M$ is top-finite. Since $\text{supp } M$ is infinite and locally finite, by König's lemma, it has a right infinite path. The proof of the corollary is completed.

We denote by $\text{Rep}^+(Q)$ and $\text{Rep}^-(Q)$ the full subcategories of $\text{Rep}(Q)$ generated by the almost finitely co-presented representations and by the almost finitely presented representations, respectively.

1.8. PROPOSITION. *The categories $\text{Rep}^+(Q)$ and $\text{Rep}^-(Q)$ are abelian and hereditary, which are closed under extensions in $\text{Rep}(Q)$.*

Proof. We shall consider only $\text{Rep}^-(Q)$. Given a morphism $f : I_1 \rightarrow I_2$ in $\text{Inj}(Q)$, set $I = \text{Im}(f)$ and consider the short exact sequence

$$(*) \quad 0 \longrightarrow I \longrightarrow I_2 \longrightarrow J \longrightarrow 0.$$

Since $\text{Rep}(Q)$ is hereditary, I, J are injective, and hence the sequence $(*)$ splits. In particular, $\text{soc } I_2 \cong \text{soc } I \oplus \text{soc } J$. Since $\text{soc } I_2$ is finitely supported and essential in I_2 , we see that $\text{soc } J$ is finitely supported and essential in J . By Corollary 1.4,

$J \in \text{Inj}(Q)$. Now it follows from the dual of Proposition 2.1 in [4] that $\text{Rep}^-(Q)$ is abelian and closed under extensions in $\text{Rep}(Q)$. Moreover, $\text{Rep}^-(Q)$ is hereditary since $\text{Rep}(Q)$ is so. The proof of the proposition is completed.

Let Σ be a subquiver of Q . For each representation $M \in \text{Rep}(Q)$, we define an object $M_\Sigma \in \text{Rep}(Q)$, called the *restriction* of M to Σ , by setting $M_\Sigma(\rho) = M(\rho)$ if $\rho \in \Sigma$; and $M_\Sigma(\rho) = 0$ otherwise, where ρ ranges over $Q_0 \cup Q_1$. For a morphism $f : M \rightarrow N$ in $\text{Rep}(Q)$, we define its *restriction to Σ* , written as $f_\Sigma : M_\Sigma \rightarrow N_\Sigma$, by setting $f_\Sigma(x) = f(x)$ if $x \in \Sigma_0$, and $f_\Sigma(x) = 0$ otherwise, where x ranges over Q_0 .

1.9. DEFINITION. Let Σ be a full subquiver of Q . A representation $M \in \text{Rep}(Q)$ is called *projective* or *injective restricted to Σ* if $M_\Sigma \in \text{Proj}(Q)$ or $M_\Sigma \in \text{Inj}(Q)$, respectively.

Let Σ be a full subquiver of Q . The *complement* of Σ in Q is the full subquiver of Q generated by the vertices not in Σ , while the *augmented complement* of Σ in Q is the full subquiver of Q generated by the vertices and the arrows not in Σ . Note that a vertex $x \in \Sigma_0$ lies in the augmented complement of Σ if and only if there exists an edge $x - y$ with $y \notin \Sigma_0$. Since Q is locally finite, the augmented complement of Σ is finite if and only if the complement is finite. We shall say that Σ is *co-finite in Q* if its complement in Q is finite.

1.10. LEMMA. *If $M \in \text{Rep}(Q)$ is injective restricted to some full subquiver Σ of Q , then it is injective restricted to any co-finite predecessor-closed subquiver of Σ .*

Proof. Let $M \in \text{Rep}(Q)$ such that $M_\Sigma \in \text{Inj}(Q)$, where Σ is a full subquiver of Q . With no loss of generality, we may assume that $M_\Sigma = I$, where $I = I_a \otimes U$ with $a \in \Sigma_0$ and U a non-zero k -space. Then $\text{supp}I \subseteq \Sigma$. Let Θ be a co-finite predecessor-closed subquiver of Σ . Then $M_\Theta = I_\Delta$, where $\Delta = \Theta \cap \text{supp}I$. Observe that Δ is co-finite and predecessor-closed in $\text{supp}I$. As a consequence, I_Δ is a quotient of I . Since $\text{Rep}(Q)$ is hereditary, I_Δ is injective. Note that $\text{supp}I$ has no right infinite path since Q is interval-finite. By Lemma 1.1, $\text{soc}I_\Delta$ is essential in I_Δ . If $a \in \Delta$, then $I_\Delta = I_a \otimes U$. Otherwise, by Lemma 1.2, for any vertex x in the support of $\text{soc}I_\Delta$, there exists an arrow $x \rightarrow y$ in $\text{supp}I$ with $y \notin \Delta$. Since Δ is co-finite in $\text{supp}I$, we see that $\text{soc}I_\Delta$ is finitely supported. By Corollary 1.4, $I_\Delta \in \text{Inj}(Q)$. That is, $M_\Theta \in \text{Inj}(Q)$. The proof of the lemma is completed.

1.11. PROPOSITION. *The intersection of $\text{Rep}^-(Q)$ and $\text{Rep}^+(Q)$ is $\text{Rep}^b(Q)$.*

Proof. Let $M \in \text{Rep}(Q)$. If M is almost finitely presented and almost finitely co-presented then, by Lemma 1.6, $\text{supp}M$ is both socle-finite and top-finite. Since Q is interval-finite, $\text{supp}M$ is finite. Conversely, assume that $\text{supp}M$ is finite. In particular, $\text{soc}M = \bigoplus_{i=1}^n (S_{a_i} \otimes U_i)$, where $a_1, \dots, a_n \in Q_0$ and U_1, \dots, U_n are non-zero k -spaces. Therefore, M has an injective envelope $M \rightarrow I$, where $I = \bigoplus_{i=1}^n (I_{a_i} \otimes U_i) \in \text{Inj}(Q)$. Consider the short exact sequence

$$(*) \quad 0 \longrightarrow M \longrightarrow I \longrightarrow J \longrightarrow 0$$

where J is injective. By Lemma 1.6, $\text{supp}I$ is socle-finite. Since Q is interval-finite, $\text{supp}I$ has no right infinite path. Since $\text{supp}J \subseteq \text{supp}I$, by Lemma 1.1, $\text{soc}J$ is essential in J . Denote by Δ the successor-closed subquiver of $\text{supp}I$ generated by $\text{supp}M$. Since $\text{supp}I$ is socle-finite, Δ is finite. Let Σ be the complement of Δ in

$\text{supp}I$. Then Σ is predecessor-closed in $\text{supp}I$. Restricting the sequence (*) to Σ yields a short exact sequence

$$0 \longrightarrow M_\Sigma \longrightarrow I_\Sigma \longrightarrow J_\Sigma \longrightarrow 0.$$

Since $M_\Sigma = 0$, we get $J_\Sigma \cong I_\Sigma$. By Lemma 1.10, $\text{soc}(J_\Sigma)$ is finite. Let x be a vertex in the support of $\text{soc}J$. If $x \in \Sigma$, then x lies in the support of $\text{soc}J_\Sigma$. Otherwise, $x \in \Delta$. This shows that $\text{soc}J$ is finitely supported. By Corollary 1.4, $J \in \text{Inj}(Q)$. That is, M is almost finitely co-presented. Dually, M is almost finitely presented. The proof of the proposition is completed.

We are ready to have a criterion for a representation to be almost finitely presented or almost finitely co-presented.

1.12. THEOREM. *Let Q be a strongly locally finite quiver. If $M \in \text{Rep}(Q)$, then*

- (1) *M is almost finitely presented if and only if M is projective restricted to some co-finite successor-closed subquiver of $\text{supp}M$, and*
- (2) *M is almost finitely co-presented if and only if M is injective restricted to some co-finite predecessor-closed subquiver of $\text{supp}M$.*

Proof. We shall prove only Statement (2). First, assume that $M_\Sigma \in \text{Inj}(Q)$, where Σ is a co-finite predecessor-closed subquiver of $\text{supp}M$. Let Ω be the complement of Σ in $\text{supp}M$. Then Ω is finite and successor-closed in $\text{supp}M$. Thus M_Ω is a finitely supported sub-representation of M , and we have a short exact sequence

$$0 \longrightarrow M_\Omega \longrightarrow M \longrightarrow M_\Sigma \longrightarrow 0$$

in $\text{Rep}(Q)$. By Propositions 1.8 and 1.11, M is almost finitely co-presented. Conversely, assume that M admits a minimal injective resolution

$$0 \longrightarrow M \xrightarrow{f} I \xrightarrow{g} J \longrightarrow 0$$

with $I = \bigoplus_{i=1}^m (I_{a_i} \otimes U_i)$ and $J = \bigoplus_{j=1}^n (I_{b_j} \otimes V_j)$, where $a_i, b_j \in Q_0$ and the U_i, V_j are k -spaces. Let Ω be the convex hull in $\text{supp}I$ generated by $a_1, \dots, a_m, b_1, \dots, b_n$. It is easy to see that Ω is finite and successor-closed in $\text{supp}I$. Let Δ be the complement of Ω in $\text{supp}I$. Then Δ is co-finite and predecessor-closed in $\text{supp}I$. We claim that the short exact sequence

$$0 \longrightarrow M_\Delta \xrightarrow{f_\Delta} I_\Delta \xrightarrow{g_\Delta} J_\Delta \longrightarrow 0$$

splits. Indeed, since $I_\Delta, J_\Delta \in \text{Inj}(Q)$ by Lemma 1.10, it suffices to show that g_Δ induces a surjective map from $\text{soc}I_\Delta$ to $\text{soc}J_\Delta$. For this purpose, fix a vertex x in the support of $\text{soc}J_\Delta$ and a non-zero element v in $(\text{soc}J_\Delta)(x)$. Since $g(x)$ is surjective, $v = g(x)(u)$ for some $u \in I(x)$. Let $\alpha_i : x \rightarrow x_i$, $i = 1, \dots, s$, be the arrows in $\text{supp}I$ starting with x , where $x_1, \dots, x_r \in \Omega$ with $0 \leq r \leq s$ and $x_{r+1}, \dots, x_s \in \Delta$. It follows from Lemma 1.2 that $I(x) = W_1 \oplus \dots \oplus W_s$ such that $I(\alpha_i)(W_j) = 0$ for $1 \leq i, j \leq s$ with $i \neq j$. Write $u = u_1 + \dots + u_s$ with $u_i \in W_i$. For $1 \leq j \leq r$, since $x_j \notin \Delta$, we have $I_\Delta(\alpha_j) = 0$, and hence $I_\Delta(\alpha_j)(u_1 + \dots + u_r) = 0$. Since $I(\alpha_j)(u_1 + \dots + u_r) = 0$ for $r < j \leq s$, we get $u_1 + \dots + u_r \in (\text{soc}I_\Delta)(x)$. If $r = s$, then $u \in (\text{soc}I_\Delta)(x)$. Otherwise, consider $g(x)(u_{r+1} + \dots + u_s) \in J(x)$. For $1 \leq j \leq r$, we have

$$J(\alpha_j)(g(x)(u_{r+1} + \dots + u_s)) = g(x_j)(I(\alpha_j)(u_{r+1} + \dots + u_s)) = g(x_j)(0) = 0.$$

For $r < j \leq s$, since $\alpha_j \in \Delta$ and $v \in \text{soc}J_\Delta$, we see that $J(\alpha_j)(v) = J_\Delta(\alpha_j)(v) = 0$. This yields

$$\begin{aligned} J(\alpha_j)(g(x)(u_{r+1} + \cdots + u_s)) &= g(x_j)(I(\alpha_j)(u_{r+1} + \cdots + u_s)) \\ &= g(x_j)(I(\alpha_j)(u_1 + \cdots + u_s)) \\ &= J(\alpha_j)(g(x)(u_1 + \cdots + u_s)) \\ &= J(\alpha_j)(v) \\ &= 0. \end{aligned}$$

Since $\text{supp}J \subseteq \text{supp}I$, we have $g(x)(u_{r+1} + \cdots + u_s) \in (\text{soc}J)(x)$, and hence $g(x)(u_{r+1} + \cdots + u_s) = 0$ since $x \neq b_j$ for $1 \leq j \leq n$. This shows that $u_1 + \cdots + u_r$ is a pre-image of v by g_Δ in $(\text{soc}I_\Delta)(x)$. Our claim is established. As a consequence, $M_\Delta \in \text{Inj}(Q)$. Finally, $\Sigma = \Delta \cap \text{supp}M$ is co-finite and predecessor-closed in $\text{supp}M$ such that $M_\Sigma = M_\Delta$. That is, M is injective restricted to Σ . The proof of the theorem is completed.

Combined with Theorem 1.12, the following result and its dual allow us to study sometimes almost finitely presented or co-presented representations of Q by means of the representation theory of finite quivers.

1.13. THEOREM. *Let Q be a strongly locally finite quiver with full subquivers $\Sigma \subseteq Q'$, and let $M, N \in \text{Rep}^-(Q)$ with $\text{supp}(M \oplus N) \subseteq Q'$ and $(M \oplus N)_\Sigma \in \text{Inj}(Q)$. If Ω is a successor-closed subquiver of Q' containing the socle-support of $(M \oplus N)_\Sigma$ and the augmented complement of Σ in Q' , then*

- (1) $M \cong N$ if and only if $M_\Omega \cong N_\Omega$;
- (2) M is indecomposable if and only if M_Ω is indecomposable;
- (3) a morphism $f : M \rightarrow N$ is a section or a retraction if and only if f_Ω is a section or a retraction, respectively.

Proof. Let Ω be a successor-closed subquiver of Q' containing the socle-support of $(M \oplus N)_\Sigma$ and the augmented complement of Σ in Q' . We first show that

$$\phi : \text{Hom}_{\text{Rep}(Q)}(M, N) \rightarrow \text{Hom}_{\text{Rep}(Q)}(M_\Omega, N_\Omega) : f \mapsto f_\Omega$$

is a k -linear isomorphism. Let $f : M \rightarrow N$ be a morphism such that $f_\Omega = 0$. If x is a vertex in the socle-support of N , then x lies in the socle-support of N_Σ or in the complement of Σ in Q' , and hence $x \in \Omega$. Thus $f(x) = 0$. Hence $\text{Im}(f) \cap \text{soc}N = 0$. Since $\text{soc}N$ is essential in N , we get $f = 0$. That is, ϕ is injective.

Next, let $\Delta = \Sigma \cap \Omega$, which is successor-closed in Σ . Hence M_Δ and N_Δ are sub-representations of M_Σ and N_Σ , respectively. Let $g : M_\Delta \rightarrow N_\Delta$ be a morphism. Consider the restriction $g_\Delta : M_\Delta \rightarrow N_\Delta$. Since N_Σ is injective, we have a morphism $h : M_\Sigma \rightarrow N_\Sigma$ such that $h_\Delta = g_\Delta$. For any $x \in Q_0$, set $f(x) = g(x)$ if $x \in \Omega$; $f(x) = h(x)$ if $x \in \Sigma$; and $f(x) = 0$ if $x \notin Q'$. Since every arrow in Q' lies in Σ or Ω , we verify easily that $f = \{f(x) \mid x \in Q_0\}$ is a morphism from M to N in $\text{Rep}(Q)$ such that $f_\Omega = g$. That is, ϕ is surjective.

Specializing to the case where $N = M$, we get $\text{End}(M) \cong \text{End}(M_\Omega)$. Since $\text{Rep}(Q)$ is abelian, an object is indecomposable if and only if its endomorphism algebra has only trivial idempotents. As a consequence, M is indecomposable if and only if M_Ω is indecomposable. This establishes Statement (2).

Finally, consider a morphism $f : M \rightarrow N$ in $\text{Rep}(Q)$. Suppose that f_Ω is a section. Let $g : N_\Omega \rightarrow M_\Omega$ be a morphism such that $gf_\Omega = 1_{M_\Omega}$. Since ϕ is surjective, there exists a morphism $h : N \rightarrow M$ such that $h_\Omega = g$. This yields

$(hf)_\Omega = (1_M)_\Omega$, and thus $hf = 1_M$. Similar, f is a retraction if f_Ω is a retraction. This establishes Statement (3). In particular, f is an isomorphism if and only if f_Ω is an isomorphism. The proof of the theorem is completed.

The rest of this section is devoted to the study of the finitely presented or co-presented representations. Let $\text{rep}^+(Q)$ stand for the category of finitely presented representations of Q and $\text{rep}^-(Q)$ for that of finitely co-presented ones.

1.14. LEMMA. *Let L, M be representations in $\text{rep}(Q)$.*

- (1) *If $M \in \text{rep}^+(Q)$, then $\text{Ext}_{\text{rep}(Q)}^i(M, L)$ is finite dimensional for $i \geq 0$.*
- (2) *If $M \in \text{rep}^-(Q)$, then $\text{Ext}_{\text{rep}(Q)}^i(L, M)$ is finite dimensional for $i \geq 0$.*

Proof. We shall prove only Statement (2). Let M be an object in $\text{rep}^-(Q)$ with a minimal injective co-resolution $0 \rightarrow M \rightarrow I \rightarrow J \rightarrow 0$, where $I, J \in \text{inj}(Q)$. Applying $\text{Hom}_{\text{rep}(Q)}(L, -)$ yields an exact sequence

$$0 \rightarrow \text{Hom}(L, M) \rightarrow \text{Hom}(L, I) \rightarrow \text{Hom}(L, J) \rightarrow \text{Ext}^1(L, M) \rightarrow 0.$$

Since $L \in \text{rep}(Q)$, by Proposition 1.3, $\text{Hom}_{\text{rep}(Q)}(L, I)$ and $\text{Hom}_{\text{rep}(Q)}(L, J)$ are finite dimensional. The proof of the lemma is completed.

One says that an additive k -category is *Hom-finite* if the Hom-spaces are of finite k -dimension and that an abelian k -category is *Ext-finite* if the Ext-spaces are of finite k -dimension. Note that a Hom-finite additive k -category is Krull-Schmidt if its idempotents split. Hence a Hom-finite abelian k -category is Krull-Schmidt. The following result is an immediate consequence of Propositions 1.8, 1.11, and 1.14.

1.15. PROPOSITION. *The k -categories $\text{rep}^+(Q)$ and $\text{rep}^-(Q)$ are Hom-finite, Ext-finite, hereditary, and abelian. Moreover, they are extension-closed in $\text{rep}(Q)$ and their intersection is $\text{rep}^b(Q)$.*

REMARK. If Q has no left infinite path or no right infinite path, then we have $\text{rep}^-(Q) = \text{rep}^b(Q)$ or $\text{rep}^+(Q) = \text{rep}^b(Q)$, respectively. As a consequence, if Q has no infinite path, then $\text{rep}^+(Q) = \text{rep}^b(Q) = \text{rep}^-(Q)$.

We shall describe the projective objects and the finite dimensional injective objects in $\text{rep}^+(Q)$. For this purpose, denote by Q^+ the full subquiver of Q generated by the vertices which are not ending point of any left infinite path.

1.16. PROPOSITION. *Let M be an indecomposable representation in $\text{rep}^+(Q)$.*

- (1) *M is a projective in $\text{rep}^+(Q)$ if and only if $M \cong P_x$ for some $x \in Q_0$.*
- (2) *M is finite dimensional and injective in $\text{rep}^+(Q)$ if and only if $M \cong I_x$ for some $x \in Q^+$.*
- (3) *If $M \cong S_x$ for some $x \in Q_0$, then M has an injective hull in $\text{rep}^+(Q)$ if and only if $x \in Q^+$.*

Proof. (1) By definition, M has a projective cover $f : P \rightarrow M$, where P is an object in $\text{proj}(Q)$. If M is projective in $\text{rep}^+(Q)$, then f is an isomorphism. Since M is indecomposable, we have $P \cong P_x$ for some $x \in Q_0$.

(2) Let $x \in Q^+$. Since Q is locally finite, x admits only finitely many predecessors in Q , and hence I_x is finite dimensional. Since I_x is injective in $\text{rep}(Q)$, it is injective in $\text{rep}^+(Q)$. Suppose conversely that M is a finite dimensional injective

object in $\text{rep}^+(Q)$. Then $\text{soc}M = \bigoplus_{i=1}^r (S_{x_i} \otimes U_i)$ with $x_i \in Q_0$ and U_i some finite dimensional non-zero k -spaces. We have an essential monomorphism $f : M \rightarrow I$ in $\text{rep}(Q)$, where $I = \bigoplus_{i=1}^r (I_{x_i} \otimes U_i)$. In particular, $\text{supp}M \subseteq \text{supp}I$. Fix a vertex y in $\text{supp}I$. Let Σ be the convex subquiver of $\text{supp}I$ generated by $\text{supp}M$ and y . Since $\text{supp}M$ is finite, so is Σ . By restriction, we get an essential monomorphism $f_\Sigma : M \rightarrow I_\Sigma$ in $\text{rep}(\Sigma)$. Observing that M is injective in $\text{rep}(\Sigma)$, we see that f_Σ is an isomorphism. In particular, $M(y) \cong I_\Sigma(y) = I(y) \neq 0$. Thus $\text{supp}M = \text{supp}I$. Therefore, f is an essential monomorphism in $\text{rep}^+(Q)$. Since M is injective in $\text{rep}^+(Q)$, f is an isomorphism, and consequently, $r = 1$ and $x_1 \in Q^+$.

(3) If $x \in Q^+$, then I_x is the injective hull of S_x in $\text{rep}^+(Q)$. Conversely, suppose that S_x has an injective hull I in $\text{rep}^+(Q)$. Since S_x is essential in I , every vertex in $\text{supp}I$ is a predecessor of x in Q . Thus $\text{supp}I$ is both top-finite and socle-finite. As a consequence, I is finite dimensional. We now deduce from Statement (2) that $I = I_x$, and hence $x \in Q^+$. The proof of the proposition is completed.

EXAMPLE. Let Q be the following infinite quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow n \longrightarrow \dots$$

Since $\text{rep}^+(Q)$ is Krull-Schmidt, using the description of the indecomposable representations in $\text{rep}^+(Q)$ given in (5.9), one can verify that P_1 is an injective object in $\text{rep}^+(Q)$. It is clear that $P_1 \notin \text{inj}(Q)$.

To conclude this section, we construct a duality between $\text{proj}(Q)$ and $\text{inj}(Q)$. The proofs will be left out since this is similar to the construction in the finite case. Consider the opposite quiver Q^{op} of Q which is defined in such a way that every vertex x in Q corresponds to a vertex x° in Q^{op} and every arrow $\alpha : x \rightarrow y$ corresponds to an arrow $\alpha^\circ : y^\circ \rightarrow x^\circ$ in Q^{op} . If $p = \alpha_n \cdots \alpha_1$ is a path in Q from x to y , then we write $p^\circ = \alpha_1^\circ \cdots \alpha_n^\circ$, the corresponding path in Q^{op} from y° to x° . For any given object $M \in \text{rep}(Q)$, we define $DM \in \text{rep}(Q^{\text{op}})$ by setting $(DM)(x^\circ) = \text{Hom}_k(M(x), k)$ for each vertex x° and $(DM)(\alpha^\circ)$ to be the transpose of $M(\alpha)$, for each arrow α° . For a morphism $f : M \rightarrow N$ in $\text{rep}(Q)$, we define a morphism $Df : DN \rightarrow DM$ in $\text{rep}(Q^{\text{op}})$ by setting $(Df)(x^\circ)$, for each vertex x° in Q^{op} , to be the transpose of $f(x)$.

1.17. LEMMA. *The functor $D : \text{rep}(Q) \rightarrow \text{rep}(Q^{\text{op}})$ is a duality such that $DI_x \cong P_{x^\circ}$ and $DP_x \cong I_{x^\circ}$, for all $x \in Q_0$.*

For the rest of the paper, put $A = kQ$, the path algebra of Q over k . Note that A has a complete set of primitive orthogonal idempotents $\{\varepsilon_x \mid x \in Q_0\}$. A left A -module M is called *unitary* if $M = \bigoplus_{x \in Q_0} \varepsilon_x M$. Let $\text{Mod}A$ be the category of left unitary A -modules. It is well known that there exists an equivalence $\text{Rep}(Q) \xrightarrow{\cong} \text{Mod}A$, sending a representation M to the module $\bigoplus_{x \in Q_0} M(x)$. For convenience, we shall make the identification $M = \bigoplus_{x \in Q_0} M(x)$. In this way, $P_x = A\varepsilon_x$ as a module, while $A = \bigoplus_{x \in Q_0} P_x$ as a representation. Note that there exists a contravariant functor $\text{Hom}_A(-, A)$ from the category of all left A -modules to that of all right A -modules which, however, does not necessarily send a unitary module to a unitary one.

1.18. LEMMA. *The functor $\text{Hom}_A(-, A) : \text{proj}(Q) \rightarrow \text{proj}(Q^{\text{op}})$ is a duality such that $\text{Hom}_A(P_x, A) \cong P_{x^\circ}$ for all $x \in Q_0$.*

Composing the dualities in Lemmas 1.17 and 1.18, we get the following result.

1.19. PROPOSITION. *The functor $\nu = D\mathrm{Hom}_A(-, A) : \mathrm{proj}(Q) \rightarrow \mathrm{inj}(Q)$, called the Nakayama functor, is an equivalence such that $\nu(P_x) \cong I_x$ for all $x \in Q_0$, whose quasi-inverse is $\nu^- = \mathrm{Hom}_{A^{\mathrm{op}}}(D-, A^{\mathrm{op}})$.*

2. ALMOST SPLIT SEQUENCES

For the rest of this paper, Q stands for a strongly locally finite quiver and k for a field. Recall that $\mathrm{rep}(Q)$ denotes the category of locally finite dimensional k -representations of Q , while $\mathrm{rep}^+(Q)$ and $\mathrm{rep}^-(Q)$ denote its full subcategories generated by the finitely presented representations and by the finitely co-presented representations, respectively. The main objective of this section is to study the Auslander-Reiten theory in $\mathrm{rep}(Q)$. Our major task is to construct an almost split sequence which ends with any given indecomposable non-projective representation in $\mathrm{rep}^+(Q)$, and one which starts with any given indecomposable non-injective representation in $\mathrm{rep}^-(Q)$. This is a more specific version of a result by Auslander; see [2, Theorem 6]. We shall also study some properties of the Auslander-Reiten translates, and show how to relate the Auslander-Reiten theory over Q to that over its subquivers.

We start with a brief recall. Let \mathcal{A} be an additive category. An object in \mathcal{A} is *strongly indecomposable* if it has a local endomorphism algebra. Let $f : X \rightarrow Y$ a morphism in \mathcal{A} . One says that f is *irreducible* if f is neither a section nor a retraction while every factorization $f = gh$ implies that h is a section or g is a retraction. Moreover, f is called *right minimal* if any morphism $h : X \rightarrow X$ such that $f = fh$ is an automorphism; *right almost split* if f is not a retraction and every non-retraction morphism $g : Z \rightarrow Y$ factors through f ; and *minimal right almost split* if f is right minimal and right almost split. In a dual manner, one defines f to be *left minimal*, *left almost split*, and *minimal left almost split*; see [6]. Note that a minimal left or right almost split morphism is irreducible if and only if it is non-zero. Furthermore, a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of morphisms in \mathcal{A} with $Y \neq 0$ is called *almost split* if f is minimal left almost split and a pseudo-kernel of g , while g is minimal right almost split and a pseudo-cokernel of f . Such an almost split sequence is unique for X and for Z if it exists, moreover, this definition coincides with the classical one in case \mathcal{A} is abelian; see [33, (1.4),(1.5)]. We say that \mathcal{A} is a *right Auslander-Reiten category* if every indecomposable object in \mathcal{A} is the co-domain of a minimal right almost split monomorphism or the ending term of an almost split sequence; a *left Auslander-Reiten category* if every indecomposable object in \mathcal{A} is the domain of a minimal left almost split epimorphism or the starting term of an almost split sequence; and an *Auslander-Reiten category* if it is left and right Auslander-Reiten; compare [33, (2.6)].

The following result is probably well known; compare [39, (I.3.2)].

2.1. LEMMA. *Let \mathcal{C} be an abelian category with a short exact sequence*

$$0 \longrightarrow X \xrightarrow{q} Y \xrightarrow{p} Z \longrightarrow 0.$$

- (1) *The morphism q is a minimal right almost split monomorphism in \mathcal{C} if and only if Z is simple and p is its projective cover.*
- (2) *The morphism p is a minimal left almost split epimorphism in \mathcal{C} if and only if X is simple and q is its injective hull.*

Proof. We shall prove only Statement (1). Suppose first that Z is simple and $p : Y \rightarrow Z$ is its projective cover. Since q is the kernel of p , it is right minimal. Let $f : M \rightarrow Y$ be a non-retraction morphism. Since Y is projective, f is not an epimorphism, and since p is superfluous, neither is pu . Since Z is simple, $pf = 0$, and hence f factors through q . That is, q is minimal right almost split.

Conversely, suppose that q is minimal right almost split. If Y is not projective, then \mathcal{C} has a non-retraction epimorphism $f : M \rightarrow Y$. Since f factors through q , we see that q is an epimorphism and hence an isomorphism, a contradiction. Thus Y is projective. If $g : L \rightarrow Y$ is not an epimorphism, then g factors through q . In particular, $pg = 0$, which is not an epimorphism. This shows that p is a superfluous epimorphism and hence a projective cover of Z . Finally, consider an arbitrary morphism $u : M \rightarrow Z$ in \mathcal{C} . Being abelian, \mathcal{C} admits a pull-back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{q'} & N & \xrightarrow{p'} & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow v & & \downarrow u & & \\ 0 & \longrightarrow & X & \xrightarrow{q} & Y & \xrightarrow{p} & Z & \longrightarrow & 0. \end{array}$$

If v is an epimorphism, then so is u . Otherwise, $v = qh$ for some $h : N \rightarrow X$, and consequently, $up' = pv = 0$. Since p' is an epimorphism, we get $u = 0$. This shows that Z is simple. The proof of the lemma is completed.

REMARK. (1) In the situation as in Lemma 2.1(1), X is the greatest sub-object of Y . One writes $X = \text{rad } Y$ and calls Z the *top* of Y .

(2) In the situation as in Lemma 2.1(2), X is the smallest sub-object of Y , which is called the *socle* of Y and written as $\text{soc } Y$.

The following statement is an immediate consequence of the preceding lemma.

2.2. COROLLARY. *If \mathcal{C} is an abelian category, then*

- (1) *\mathcal{C} is right Auslander-Reiten if and only if every indecomposable non-projective object is the ending term of an almost split sequence and every indecomposable projective object has a simple top;*
- (2) *\mathcal{C} is left Auslander-Reiten if and only if every indecomposable non-injective object is the starting term of an almost split sequence and every indecomposable injective object has a simple socle.*

We now begin to study the Auslander-Reiten theory in $\text{rep}(Q)$. Note that the indecomposable objects $\text{rep}(Q)$ are all strongly indecomposable; see [19, (3.6)]. The following result, which is an immediate consequence of Lemma 2.1, will be used frequently.

2.3. LEMMA. *If $x \in Q_0$, then the inclusion $q_x : \text{rad } P_x \rightarrow P_x$ is a minimal right almost split monomorphism, while the projection $p_x : I_x \rightarrow I_x / \text{soc } I_x$ is a minimal left almost split epimorphism in $\text{rep}(Q)$.*

The following construction is analogous to the classical one for modules over an artin algebra; see, for example, [7].

2.4. DEFINITION. Let M be a representation in $\text{rep}(Q)$.

- (1) If M has a minimal projective resolution $0 \rightarrow P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ with $P_1, P_0 \in \text{proj}(Q)$, then $\text{DTr}M$ denotes the kernel of $\nu(f) : \nu(P_1) \rightarrow \nu(P_0)$.
- (2) If M has a minimal injective co-resolution $0 \rightarrow M \rightarrow I_0 \xrightarrow{g} I_1 \rightarrow 0$ with $I_0, I_1 \in \text{inj}(Q)$, then $\text{Tr}DM$ denotes the co-kernel of $\nu^-(g) : \nu^-(I_0) \rightarrow \nu^-(I_1)$.

REMARK. (1) $\text{DTr}M$ is defined only up to isomorphism and only for $M \in \text{rep}^+(Q)$, in such a way that $\text{DTr}M = 0$ if and only if $M \in \text{proj}(Q)$.

(2) $\text{Tr}DM$ is defined only up to isomorphism and only for $M \in \text{rep}^-(Q)$, in such a way that $\text{Tr}DM = 0$ if and only if $M \in \text{inj}(Q)$.

The following lemma and its dual play an important role in the construction of almost split sequences.

2.5. LEMMA. *Let $M \in \text{rep}^+(Q)$ be indecomposable with a minimal projective resolution*

$$0 \rightarrow P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0.$$

If M is not projective, then $\text{DTr}M \cong \text{DExt}_A^1(M, A) \in \text{rep}^-(Q)$, which is indecomposable and not injective with a minimal injective co-resolution

$$0 \rightarrow \text{DTr}M \rightarrow \nu(P_1) \xrightarrow{\nu(f)} \nu(P_0) \rightarrow 0.$$

Proof. Suppose that M is not projective. Since $\text{Rep}(Q)$ is hereditary, we have $\text{Hom}_A(M, A) = 0$. Applying $\text{Hom}_A(-, A)$ to the minimal projective resolution stated in the lemma, we get a short exact sequence of right A -modules as follows:

$$0 \rightarrow \text{Hom}_A(P_0, A) \xrightarrow{f^*} \text{Hom}_A(P_1, A) \rightarrow \text{Ext}_A^1(M, A) \rightarrow 0,$$

where $\text{Ext}_A^1(M, A)$ is unitary since $\text{Hom}_A(P_0, A)$ and $\text{Hom}_A(P_1, A)$ are unitary. Applying the duality $D : \text{rep}(Q^{\text{op}}) \rightarrow \text{rep}(Q)$ stated in Lemma 1.17, we obtain a short exact sequence

$$\eta : 0 \rightarrow \text{DExt}_A^1(M, A) \rightarrow \nu(P_1) \xrightarrow{\nu(f)} \nu(P_0) \rightarrow 0$$

in $\text{rep}(Q)$. By definition, $\text{DTr}M \cong \text{DExt}_A^1(M, A) \in \text{rep}^-(Q)$. Furthermore, since $\text{Im}(f) \subseteq \text{rad}P_0$, we see that $\text{Im}(f^*)$ is contained in the radical of $\text{Hom}_A(P_1, A)$, and hence the kernel of $\nu(f)$ contains the socle of $\nu(P_1)$. That is, η is a minimal injective co-resolution of $\text{DTr}M$. In particular, $\text{DTr}M$ is not injective. Finally, since ν is an equivalence and M is indecomposable, $\text{DTr}M$ is indecomposable. The proof of the lemma is completed.

As a consequence of Lemma 2.5 and its dual, we have the following result.

2.6. COROLLARY. *If M, N are indecomposable objects in $\text{rep}(Q)$, then $N \cong \text{DTr}M$ if and only if $M \cong \text{Tr}DN$.*

The following consequence of Lemma 2.5 will be needed later; compare [6, (4.2)].

2.7. COROLLARY. *Let $M, N \in \text{rep}^+(Q)$ be indecomposable and not projective. If $\text{rep}^+(Q)$ has a monomorphism $f : M \rightarrow N$, then $\text{rep}^-(Q)$ has a monomorphism $g : \text{DTr}M \rightarrow \text{DTr}N$.*

Proof. Since $\text{rep}(Q)$ is hereditary, $D\text{Ext}_A^1(-, A) : \text{rep}^+(Q) \rightarrow \text{rep}^-(Q)$ is a left exact functor. If $f : M \rightarrow N$ is a monomorphism in $\text{rep}^+(Q)$, then

$$g = D\text{Ext}_A^1(f, A) : D\text{Ext}_A^1(M, A) \rightarrow D\text{Ext}_A^1(N, A)$$

is a monomorphism in $\text{rep}^-(Q)$. By Lemma 2.5, g is a monomorphism from $D\text{Tr}M$ to $D\text{Tr}N$. The proof of the corollary is completed.

We are ready to have the existence theorem for almost split sequences.

2.8. THEOREM. *Let Q be a strongly locally finite quiver, and let $M \in \text{rep}(Q)$ be indecomposable.*

- (1) *If $M \in \text{rep}^+(Q)$ is not projective, then $\text{rep}(Q)$ admits an almost split sequence $0 \rightarrow D\text{Tr}M \rightarrow N \rightarrow M \rightarrow 0$, where $D\text{Tr}M \in \text{rep}^-(Q)$.*
- (2) *If $M \in \text{rep}^-(Q)$ is not injective, then $\text{rep}(Q)$ admits an almost split sequence $0 \rightarrow M \rightarrow N \rightarrow \text{Tr}DM \rightarrow 0$, where $\text{Tr}DM \in \text{rep}^+(Q)$.*

Proof. We only prove Statement (1). Assume that M is finitely presented and not projective. By Lemma 2.5, $D\text{Tr}M$ is finitely co-presented, indecomposable, and not injective. Let $L \in \text{rep}(Q)$. By Lemma 1.14, $\text{Ext}_A^1(M, L)$ and $\text{Hom}_A(L, D\text{Tr}M)$ are of finite k -dimension. We claim, for $P \in \text{proj}(Q)$, that there exists a k -linear isomorphism, which is natural in P and L , as follows:

$$\psi_{L,P} : \text{Hom}_A(P, L) \rightarrow D\text{Hom}_A(L, \nu P).$$

Indeed, we may assume with no loss of generality that $P = P_x$ for some $x \in Q_0$. By Proposition 1.3, we have the following k -isomorphisms:

$$\text{Hom}_A(P_x, L) \cong L(x) \cong D\text{Hom}_k(L(x), k) \cong D\text{Hom}_A(L, I_x),$$

each of which is natural in P_x and L . This establishes our claim.

Let $0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a minimal projective resolution of M , where $P_0, P_1 \in \text{proj}(Q)$. By Lemma 2.5, $D\text{Tr}M$ has a minimal injective co-resolution $0 \rightarrow D\text{Tr}M \rightarrow \nu(P_1) \rightarrow \nu(P_0) \rightarrow 0$, where $\nu(P_1), \nu(P_0) \in \text{inj}(Q)$. Applying $\text{Hom}_A(-, L)$ and $D\text{Hom}_A(L, -)$, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \text{Hom}_A(P_0, L) & \longrightarrow & \text{Hom}_A(P_1, L) & \longrightarrow & \text{Ext}_A^1(M, L) & \longrightarrow & 0 \\ \downarrow \psi_{P_0, L} & & \downarrow \psi_{P_1, L} & & \downarrow \phi_L & & \\ D\text{Hom}_A(L, \nu P_0) & \longrightarrow & D\text{Hom}_A(L, \nu P_1) & \longrightarrow & D\text{Hom}_A(L, D\text{Tr}M) & \longrightarrow & 0, \end{array}$$

where $\psi_{P_0, L}$ and $\psi_{P_1, L}$ are natural isomorphisms. Thus there exists an isomorphism $\phi_L : \text{Ext}_A^1(M, L) \rightarrow D\text{Hom}_A(L, D\text{Tr}M)$, which is natural in L . Since $\text{End}(D\text{Tr}M)$ is finite dimensional, there exists a non-zero k -linear map $\theta : \text{End}(D\text{Tr}M) \rightarrow k$, which vanishes on $\text{rad}(\text{End}(D\text{Tr}M))$. Consider the corresponding non-zero element

$$\eta = \phi_{D\text{Tr}M}^{-1}(\theta) : 0 \rightarrow D\text{Tr}M \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0$$

in $\text{Ext}^1(M, D\text{Tr}M)$. Let $u : D\text{Tr}M \rightarrow L$ be a non-section morphism in $\text{rep}(Q)$. For any $v : L \rightarrow D\text{Tr}M$, since $vu \in \text{rad}(\text{End}(D\text{Tr}M))$, we have $\theta(vu) = 0$. This shows that $D\text{Hom}(u, D\text{Tr}M)(\theta) = 0$. In view of the commutative diagram

$$\begin{array}{ccc}
 \text{Ext}^1(M, \text{DTr} M) & \xrightarrow{\text{Ext}^1(M, u)} & \text{Ext}^1(M, L) \\
 \downarrow \phi_{\text{DTr} M} & & \downarrow \phi_L \\
 \text{DEnd}(\text{DTr} M) & \xrightarrow{\text{DHom}(u, \text{DTr} M)} & \text{DHom}(L, \text{DTr} M),
 \end{array}$$

we get $\text{Ext}^1(M, u)(\eta) = 0$, that is, u factors through f . Thus η is an almost split sequence in $\text{rep}(Q)$; see [6, (2.14)]. The proof of the theorem is completed.

REMARK. It is shown in [38] that every almost split sequence in $\text{rep}(Q)$ is of the form stated in Theorem 2.8.

The following result is a consequence of Theorem 2.8 and Proposition 1.15; compare [15, (1.20)] and [14, Theorem 4].

2.9. COROLLARY. *Let M be an indecomposable representation in $\text{rep}^b(Q)$.*

- (1) *If M is not projective, then $\text{rep}(Q)$ has an almost split sequence ending with M , which is also an almost split sequence in $\text{rep}^-(Q)$.*
- (2) *If M is not injective, then $\text{rep}(Q)$ has an almost split sequence starting with M , which is also an almost split sequence in $\text{rep}^+(Q)$.*

Next, we shall study the Auslander-Reiten translates. To this end, the following easy result is useful.

2.10. LEMMA. *Let $M \in \text{rep}^+(Q)$ be indecomposable with a minimal projective resolution*

$$0 \longrightarrow \bigoplus_{j=1}^s P_{y_j} \longrightarrow \bigoplus_{i=1}^r P_{x_i} \xrightarrow{f} M \longrightarrow 0.$$

If $x \in Q_0$ is not in $\text{supp} M$, then $x = y_j$ for some $1 \leq j \leq s$ if and only if x is an immediate successor of some vertex in $\text{supp} M$.

Proof. Let N denote the kernel of f . Then y_1, \dots, y_s are the vertices in the top-support of N . Fix a vertex x not lying in $\text{supp} M$. Suppose first that Q has an arrow $\alpha : y \rightarrow x$ with $y \in \text{supp} M$. Since f is surjective, there exists a path p in Q from some x_i to y such that $f(p) \neq 0$. Since $x \notin \text{supp} M$, we have $f(\alpha p) = 0$. Thus αp lies in N but not in its radical, and hence, x lies in the top-support of N .

Suppose conversely that x lies in the top-support of N . Then there exists some $\rho = \lambda_1 p_1 + \dots + \lambda_t p_t$, where $\lambda_1, \dots, \lambda_t \in k$, and p_1, \dots, p_t are paths in Q from some of x_1, \dots, x_r to x , which lies in N but not in its radical. Since $x \notin \text{supp} M$, we may write $p_i = \alpha_i q_i$, where α_i is an arrow ending in x and q_i is a path in Q , for $i = 1, \dots, t$. Since $\rho \notin \text{rad} N$, there exists some $1 \leq i_0 \leq t$ such that $f(q_{i_0}) \neq 0$. In particular, $e(q_{i_0}) \in \text{supp} M$. The proof of the lemma is completed.

The following result is an immediate consequence of Lemmas 2.5 and 2.10.

2.11. COROLLARY. *Let $M \in \text{rep}^+(Q)$ be indecomposable. If x is a vertex in Q not lying in $\text{supp} M$, then x lies in the socle-support of $\text{DTr} M$ if and only if x is an immediate successor of some vertex in $\text{supp} M$.*

Let M be an indecomposable representation in $\text{rep}(Q)$. By convention, we write $\text{DTr}^0M = M = \text{TrD}^0M$. If $M \in \text{rep}^+(Q)$, then $\text{DTr}M \in \text{rep}^-(Q)$; and if, moreover, $\text{DTr}M \in \text{rep}^+(Q)$, then DTr^2M is defined and lies in $\text{rep}^-(Q)$. In general, if $n > 0$ is such that $\text{DTr}^{n-1}M$ is defined and lies in $\text{rep}^+(Q)$, then DTr^nM is defined and lies in $\text{rep}^-(Q)$. If DTr^nM is defined and non-zero for some $n > 0$, then it follows from Proposition 1.15 and Lemma 2.5 that DTr^iM is indecomposable for $0 \leq i \leq n$, and finite dimensional for $0 < i < n$. We shall say that M is *DTr-stable* if DTr^nM is defined and non-zero for all $n \geq 0$, or equivalently, DTr^nM is indecomposable of finite dimension for all $n > 0$.

2.12. LEMMA. *Let $M \in \text{rep}^+(Q)$ be indecomposable, and let w be an infinite acyclic walk in Q which starts with an arrow and intersects $\text{supp}M$ only at $s(w)$. Then M is DTr-stable or DTr^nM is infinite dimensional for some $n \geq 0$. Furthermore,*

- (1) *If all but finitely many edges in w are inverses of arrows, then DTr^nM is infinite dimensional for some $n \geq 0$.*
- (2) *If DTr^mM with $m > 0$ is defined, then its support contains some vertex lying in w but different from $s(w)$.*

Proof. Write $w = \cdots w_n \cdots w_2 w_1$, where the w_i are edges. Put $a_i = s(w_i)$ for $i \geq 1$. Set $s_0 = 1$. Then w_{s_0} is an arrow and s_0 is maximal such that $a_{s_0} \in \text{supp}M$. Let $r \geq 0$ be an integer such that $\text{DTr}^rM \in \text{rep}^+(Q)$ and there exist integers $s_0 < \cdots < s_r$ satisfying the following property: w_{s_i} is an arrow and s_i is maximal for which a_{s_i} is in the support of DTr^iM , for $i = 0, \dots, r$. Since w_{s_r} is an arrow $a_{s_r} \rightarrow a_{s_r+1}$ and a_{s_r+1} is not in the support of DTr^rM , by Corollary 2.11, a_{s_r+1} is in the support of $\text{DTr}^{r+1}M$. If $\text{DTr}^{r+1}M \notin \text{rep}^+(Q)$, then $\text{DTr}^{r+1}M$ is infinite dimensional and DTr^iM is not defined for every $i > r+1$. In this case, the lemma is proved and we stop the process. Otherwise, $\text{DTr}^{r+1}M$ is non-zero of finite dimension by Proposition 1.15. Therefore, there exists a maximal integer $s_{r+1} > s_r$ such that $a_{s_{r+1}}$ is in the support of $\text{DTr}^{r+1}M$. Suppose that $w_{s_{r+1}}$ is the inverse of an arrow $a_{s_{r+1}} \leftarrow a_{s_{r+1}+1}$. Since $a_{s_{r+1}+1}$ is not in the support of $\text{DTr}^{r+1}M$, applying the dual of Corollary 2.11 to $\text{DTr}^{r+1}M$, we see that $a_{s_{r+1}+1}$ is in the support of DTr^rM , contrary to the maximality of s_r . Therefore, $w_{s_{r+1}}$ is an arrow $a_{s_{r+1}} \rightarrow a_{s_{r+1}+1}$. If this process never stops, then we get an infinite increasing sequence of integers

$$s_0 < s_1 < \cdots < s_i < \cdots$$

satisfying the above-stated property. In particular, M is DTr-stable, and a_{s_i} lies in the support of DTr^iM for all $i \geq 0$. Moreover, since the w_{s_i} are arrows, the hypothesis stated in Statement (1) does not occur. The proof of the lemma is completed.

REMARK. If Q has no left infinite path, then $\text{rep}^-(Q) = \text{rep}^b(Q)$. In this case, Lemma 2.12 provides a simple combinatorial condition for an indecomposable representation to be DTr-stable.

Dually, if $M \in \text{rep}^-(Q)$ is indecomposable, then $\text{TrD}M \in \text{rep}^+(Q)$. If $n > 0$ is such that $\text{TrD}^{n-1}M$ is defined and lies in $\text{rep}^-(Q)$, then TrD^nM is defined and lies in $\text{rep}^+(Q)$. If TrD^nM is defined and non-zero for some $n > 0$, then it follows from Proposition 1.15 and the dual of Lemma 2.5 that TrD^iM is indecomposable for $0 \leq i \leq n$, and finite dimensional for $0 < i < n$. We shall say that M is *TrD-stable* if TrD^nM is defined and non-zero for all $n \geq 0$, or equivalently, TrD^nM

is indecomposable of finite dimension for all $n > 0$. The following result is a dual statement of Lemma 2.12.

2.13. LEMMA. *Let $M \in \text{rep}^-(Q)$ be indecomposable, and let w be an infinite acyclic walk in Q which ends with an arrow and intersects $\text{supp}M$ only at $e(w)$. Then M is TrD -stable or $\text{TrD}^n M$ is infinite dimensional for some $n \geq 0$. Moreover,*

- (1) *If all but finitely many edges in w are inverses of arrows, then $\text{TrD}^n M$ is infinite dimensional for some $n \geq 0$.*
- (2) *If $\text{TrD}^m M$ with $m > 0$ is defined, then its support contains some vertex lying in w but different from $e(w)$.*

The preceding results yield some very useful consequences.

2.14. PROPOSITION. *Let Q be an infinite connected strongly locally finite quiver.*

- (1) *For any $x, y \in Q_0$, there exists no integer $m \geq 0$ such that $\text{DTr}^m I_y \cong P_x$.*
- (2) *If $M, N \in \text{rep}(Q)$ are indecomposable such that $M \cong \text{DTr}^n N$ for some $n \geq 0$, then $\text{supp}M = \text{supp}N$ if and only if $n = 0$.*

Proof. (1) Let $x, y \in Q_0$ be such that $P_x \cong \text{DTr}^m I_y$ for some $m \geq 0$. If $m = 0$, then it is easy to see that Q consists of a single path from x to y , a contradiction. Thus $m > 0$. Since $P_x \in \text{rep}^+(Q)$, by Proposition 1.15, $\text{DTr}^i I_y$ is finite dimensional, for $i = 1, \dots, m$. On the other hand, since $I_y = \text{TrD}^m P_x$ by Corollary 2.6, I_y is finite dimensional by Proposition 1.15. Since $\text{DTr}^{m+1} I_y = 0$, we see that I_y is not DTr -stable and $\text{DTr}^i I_y$ is finite dimensional for all $i \geq 0$. Since Q is connected and infinite, applying König's lemma to the complement of $\text{supp}I_y$, we get a right infinite acyclic walk w which intersects $\text{supp}I_y$ only at $s(w)$. Since $\text{supp}I_y$ is successor-closed, w starts with an arrow. By Lemma 2.12, I_y is DTr -stable or $\text{DTr}^r I_y$ is infinite dimensional for some $r \geq 0$, a contradiction.

(2) Let $M, N \in \text{rep}(Q)$ be indecomposable such that $M \cong \text{DTr}^n N$ with $n \geq 0$. By Corollary 2.6, $N \cong \text{TrD}^n M$. Suppose that $n > 0$ and that $\text{supp}M = \text{supp}N = \Sigma$. Then $N \in \text{rep}^+(Q)$ and $M \in \text{rep}^-(Q)$. Moreover, by Lemma 1.6, Σ is top-finite and socle-finite, and hence finite. Applying König's lemma to the complement of Σ , we get a left infinite acyclic walk w which intersects Σ only at $e(w)$. If w ends with an arrow, then we may apply Lemma 2.13(2) to M to see that $\text{supp}N$, that is Σ , contains some vertex lying in w but different from $e(w)$, a contradiction. If w ends with the inverse of an arrow, then w^{-1} is an infinite acyclic walk which starts with an arrow and intersects $\text{supp}N$ only at $s(w^{-1})$. It follows from Lemma 2.12(2) that $\text{supp}M$, that is Σ , contains some vertex lying in w^{-1} but different from $s(w^{-1})$, a contradiction again. The proof of the proposition is completed.

REMARK. Proposition 2.14(1) is well known in the finite non-Dynkin case. If Q is infinite without left infinite paths, Reiten and Van der Bergh proved this by using a highly indirect argument to treat the infinite Dynkin case; see [39].

Let Σ be a subquiver of Q . If M is a representation of Q supported by Σ , for the sake of convenience, we shall regard M as a representation of Σ whenever no risk of confusion is possible. In particular, if N is a representation of Q , then its restriction N_Σ will be regarded as a representation of Σ . On the other hand, every representation M of Σ can be extended trivially to a representation of Q which, by abuse of notation, is denoted again by M . In this way, we shall identify $\text{rep}(\Sigma)$

with the full subcategory of $\text{rep}(Q)$ generated by the representations supported by Σ . As one of the techniques for our later investigation, we shall relate the almost split sequences and the irreducible morphisms in $\text{rep}(Q)$ to those in $\text{rep}(\Sigma)$.

2.15. LEMMA. *Let Σ be a convex subquiver of Q , and let $M \in \text{rep}^+(Q)$.*

- (1) *If Σ is predecessor-closed in Q , then $M_\Sigma \in \text{rep}^+(\Sigma)$.*
- (2) *If Σ contains the trivial and the immediate successors of the vertices in $\text{supp}M$, then $M \in \text{rep}^+(\Sigma)$.*

Proof. For $x \in Q_0$, let P'_x denote the restriction of P_x to Σ . Since Σ is convex, for $x \in \Sigma$, it is easy to see that P'_x is isomorphic to the indecomposable projective representation in $\text{rep}(\Sigma)$ at x . Now, M has a minimal projective resolution

$$\eta : \quad 0 \longrightarrow \bigoplus_{j=1}^s P_{y_j} \longrightarrow \bigoplus_{i=1}^r P_{x_i} \longrightarrow M \longrightarrow 0.$$

Restricting η to Σ , we get a short exact sequence in $\text{rep}(\Sigma)$ as follows :

$$\eta_\Sigma : \quad 0 \longrightarrow \bigoplus_{j=1}^s P'_{y_j} \longrightarrow \bigoplus_{i=1}^r P'_{x_i} \longrightarrow M_\Sigma \longrightarrow 0.$$

Suppose that Q is predecessor-closed in Q . Then $P'_x = 0$, for $x \notin \Sigma$. This implies that $P'_x \in \text{proj}(\Sigma)$, for all $x \in Q_0$. In particular, η_Σ is a minimal projective resolution of M_Σ in $\text{rep}(\Sigma)$. That is, $M_\Sigma \in \text{rep}^+(\Sigma)$.

Suppose next that Σ contains the trivial and the immediate successors of the vertices in $\text{supp}M$. Then $M_\Sigma = M$, and by Corollary 2.11, the x_i and the y_j all lie in Σ . Hence, the P'_{x_i} and the P'_{y_j} all lie in $\text{proj}(\Sigma)$. As a consequence, η_Σ is a minimal projective resolution of M in $\text{rep}(\Sigma)$. That is, $M \in \text{rep}^+(\Sigma)$. The proof of the lemma is completed.

2.16. PROPOSITION. *Let $N \in \text{rep}^+(Q)$ be indecomposable, and let Σ be a convex subquiver of Q containing all the predecessors of the trivial and the immediate successors of the vertices in $\text{supp}N$.*

- (1) *The almost split sequence in $\text{rep}(\Sigma)$ ending with N is almost split in $\text{rep}(Q)$.*
- (2) *Any irreducible morphism in $\text{rep}(\Sigma)$ ending in N is irreducible in $\text{rep}(Q)$.*

Proof. First of all, N can be considered to be a representation of Σ , which is finitely presented by Lemma 2.15(2). Moreover, since Σ contains the immediate successors of the vertices in $\text{supp}N$, we see that N is projective in $\text{rep}(\Sigma)$ if and only if it is projective in $\text{rep}(Q)$.

(1) Let $\eta : 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an almost split sequence in $\text{rep}(\Sigma)$. In particular, N is not projective in $\text{rep}(\Sigma)$, and hence not projective in $\text{rep}(Q)$. Then, $\text{rep}(Q)$ has an almost split sequence $\zeta : 0 \longrightarrow \text{DTr}N \longrightarrow E \longrightarrow N \longrightarrow 0$. By Corollary 2.11, the vertices in the socle-support of $\text{DTr}N$ are trivial or immediate successors of the vertices in $\text{supp}N$. Since the socle of $\text{DTr}N$ is essential, the vertices in the support of $\text{DTr}N$ are all predecessors of the vertices in the socle-support of $\text{DTr}N$, which lie in Σ by the hypothesis stated in the proposition. Therefore, ζ lies entirely in $\text{rep}(\Sigma)$. Then ζ is an almost split sequence in $\text{rep}(\Sigma)$, and hence, it is isomorphic to η . In other words, η is almost split in $\text{rep}(Q)$.

(2) Let $f : M \rightarrow N$ be an irreducible morphism in $\text{rep}(\Sigma)$. Since $N \in \text{rep}^+(\Sigma)$, by Lemma 2.3 and Theorem 2.8(1), $\text{rep}(\Sigma)$ has a minimal right almost split morphism $g : L \rightarrow N$. Then $f = gs$ for some section $s : M \rightarrow L$. If N is not projective in $\text{rep}(\Sigma)$, then g is minimal right almost split in $\text{rep}(Q)$ by Statement (1). Otherwise,

$N = P_x$ for some $x \in \Sigma_0$, and g is the inclusion map $\text{rad}P_x \rightarrow P_x$. In any case, f is irreducible in $\text{rep}(Q)$. The proof of the proposition is completed.

Conversely, we have the following result; compare [14, Lemma 2].

2.17. PROPOSITION. *Let $N \in \text{rep}^+(Q)$ be indecomposable, and let Σ be a convex subquiver of Q containing the trivial and the immediate successors of the vertices in $\text{supp}N$.*

- (1) *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an almost split sequence in $\text{rep}(Q)$, then its restriction $0 \rightarrow L_\Sigma \rightarrow M_\Sigma \rightarrow N \rightarrow 0$ is an almost split sequence in $\text{rep}(\Sigma)$.*
- (2) *If $f : M \rightarrow N$ is an irreducible morphism in $\text{rep}(Q)$, then its restriction $f_\Sigma : M_\Sigma \rightarrow N$ is an irreducible morphism in $\text{rep}(\Sigma)$.*

Proof. (1) Consider an almost split sequence $\xi : 0 \rightarrow L \rightarrow M \xrightarrow{g} N \rightarrow 0$ in $\text{rep}(Q)$. Restricting ξ to Σ yields a short exact sequence

$$\xi_\Sigma : 0 \rightarrow L_\Sigma \rightarrow M_\Sigma \xrightarrow{g_\Sigma} N \rightarrow 0$$

in $\text{rep}(\Sigma)$. If g_Σ is a retraction, then $\text{rep}(\Sigma)$ has a morphism $h' : N \rightarrow M_\Sigma$ such that $g_\Sigma h' = \mathbf{1}_N$. Since Σ contains the immediate successors of the vertices in $\text{supp}N$, we can extend h' to a morphism $h : N \rightarrow M$ in $\text{rep}(Q)$ such that $gh = \mathbf{1}_N$, a contradiction. If $u : X \rightarrow N$ is a non-retraction morphism in $\text{rep}(\Sigma)$, then it is not a retraction in $\text{rep}(Q)$. Thus $u = gv$, for some morphism $v : X \rightarrow M$ in $\text{rep}(Q)$. Restricting the equation to Σ yields $u = g_\Sigma v_\Sigma$. This shows that g_Σ is right almost split in $\text{rep}(\Sigma)$.

For $x \in Q_0$, let P'_x and I'_x denote the restrictions of P_x and I_x to Σ , respectively. If $x \in \Sigma$ then, since Σ is convex, P'_x and I'_x are isomorphic to the indecomposable projective and injective representations in $\text{rep}(\Sigma)$ at x , respectively. Now, N has a minimal projective resolution

$$\eta : 0 \rightarrow \bigoplus_{j=1}^s P_{y_j} \xrightarrow{w} \bigoplus_{i=1}^r P_{x_i} \rightarrow N \rightarrow 0$$

in $\text{rep}(Q)$. By Corollary 2.11, the x_i and the y_j all lie in Σ . Thus, restricting η to Σ , we get a minimal projective resolution

$$\eta_\Sigma : 0 \rightarrow \bigoplus_{j=1}^s P'_{y_j} \xrightarrow{w_\Sigma} \bigoplus_{i=1}^r P'_{x_i} \rightarrow N \rightarrow 0$$

of N in $\text{rep}(\Sigma)$. On the other hand, by Lemma 2.5, L has a minimal injective co-resolution

$$\zeta : 0 \rightarrow L \rightarrow \bigoplus_{j=1}^s I_{y_j} \xrightarrow{\nu(w)} \bigoplus_{i=1}^r I_{x_i} \rightarrow 0$$

in $\text{rep}(Q)$. Restricting ζ to Σ , we obtain a minimal injective co-resolution

$$\zeta_\Sigma : 0 \rightarrow L_\Sigma \rightarrow \bigoplus_{j=1}^s I'_{y_j} \xrightarrow{\nu(w)_\Sigma} \bigoplus_{i=1}^r I'_{x_i} \rightarrow 0$$

of L_Σ in $\text{rep}(\Sigma)$. Moreover, it follows from the definition that $\nu(w)_\Sigma = \nu_\Sigma(w_\Sigma)$, where ν_Σ is the Nakayama functor for $\text{rep}(\Sigma)$. This implies that $L_\Sigma = D\text{Tr}_\Sigma M$. By Lemma 2.5, L_Σ is indecomposable and hence strongly indecomposable. Thus, ξ_Σ is an almost split sequence in $\text{rep}(\Sigma)$; see [6, (2.14)].

(2) Assume that $f : M \rightarrow N$ is an irreducible morphism in $\text{rep}(Q)$. By Lemma 2.3 and Theorem 2.8, $\text{rep}(Q)$ has a minimal right almost split morphism $g : L \rightarrow N$. Then $f = gs$, where $s : M \rightarrow L$ is a section. Hence, $f_\Sigma = g_\Sigma s_\Sigma$, where s_Σ is clearly

a section. If N is not projective, then g_x is minimal right almost split in $\text{rep}(\Sigma)$ by Statement (1). Otherwise, $N = P_x$ and $L = \text{rad}P_x$ for some $x \in Q_0$. By the hypothesis, both L and N are supported by Σ . So $g_x = g$, which is minimal right almost split in $\text{rep}(\Sigma)$. In any case, f_x is irreducible in $\text{rep}(\Sigma)$. The proof of the proposition is completed.

3. AUSLANDER-REITEN CATEGORIES

In the following four sections, we shall be mainly concerned with the study of Auslander-Reiten theory in $\text{rep}^+(Q)$. It is left to the reader to formulate the dual results for $\text{rep}^-(Q)$. In case Q has no left infinite path, Reiten and Van den Bergh proved that $\text{rep}^+(Q)$ is a right Auslander-Reiten category; see [39]. The main objective of this section is to find the necessary and sufficient conditions for $\text{rep}^+(Q)$ to be a left or a right Auslander-Reiten category.

We begin with studying some properties of irreducible morphisms in $\text{rep}^+(Q)$.

3.1. LEMMA. *If $f : M \rightarrow N$ is an irreducible epimorphism in $\text{rep}^+(Q)$, then the kernel of f is finite dimensional.*

Proof. Let $f : M \rightarrow N$ be an irreducible epimorphism in $\text{rep}^+(Q)$. Since $\text{rep}^+(Q)$ is Krull-Schmidt, we may assume that N is indecomposable; see [9, (3.1), (3.2)]. Since N is not projective, by Theorem 2.8(1), there exists in $\text{rep}(Q)$ an almost split sequence $\eta : 0 \rightarrow L \rightarrow E \xrightarrow{g} N \rightarrow 0$, where $L \in \text{rep}^-(Q)$. This yields a push-out diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & M & \xrightarrow{f} & N \longrightarrow 0 \\ & & \downarrow v & & \downarrow u & & \parallel \\ 0 & \longrightarrow & L & \longrightarrow & E & \xrightarrow{g} & N \longrightarrow 0. \end{array}$$

Set $\Sigma = \text{supp}M$ and $\Omega = \text{supp}E$. Let Θ be the full subquiver of Ω generated by the vertices which are successors in Q of the vertices in Σ . Since Σ is top-finite by Lemma 1.6 and Q is interval-finite, Θ has no left infinite path. Since $\text{supp}N \subseteq \Sigma \cap \Omega \subseteq \Theta$, restricting η to Θ yields a short exact sequence

$$\eta_\Theta : 0 \longrightarrow L_\Theta \longrightarrow E_\Theta \xrightarrow{g_\Theta} N \longrightarrow 0.$$

Suppose that $\text{supp}L_\Theta$, that is $\Theta \cap \text{supp}L$, contains infinitely many vertices x_i , $i \in \mathbb{N}$. Since $\text{supp}L$ is socle-finite by Lemma 1.6, we may assume that $\text{supp}L$ contains a path $p_i : x_i \rightsquigarrow a$, for each $i \in \mathbb{N}$, where a is some fixed vertex in $\text{supp}L$. Since $\text{supp}L \subseteq \Omega$ and Θ is successor-closed in Ω , the p_i all lie in Θ . Being locally finite, by König's lemma, Θ has a left infinite path ending with a , a contradiction. Thus $L_\Theta \in \text{rep}^b(Q)$, and consequently, $E_\Theta \in \text{rep}^+(Q)$. Note that E_Θ is a subrepresentation of E since Θ is successor-closed in Ω . Moreover, the support of $\text{Im}(u)$ is contained in $\Sigma \cap \Omega \subseteq \Theta$. Thus $u = qu'$, where $u' : M \rightarrow E_\Theta$ is the co-restriction of u , and $q : E_\Theta \rightarrow E$ is the inclusion. This yields a factorisation $f = (gq)u'$ in $\text{rep}^+(Q)$. Thus gq is a retraction or u' is a section. Since g is not a retraction, the first case does not occur. In particular, $u = qu'$ is a monomorphism, and so is v . Since $\text{supp}L$ has no right infinite path by Lemma 1.6, nor does $\text{supp}X$. On the other hand, $X \in \text{rep}^+(Q)$ since it is the kernel of f . By Corollary 1.7, $X \in \text{rep}^b(Q)$. The proof of the lemma is completed.

3.2. LEMMA. *Let $f : M \rightarrow N$ be an irreducible morphism in $\text{rep}^+(Q)$. If M is infinite dimensional, then N is infinite dimensional while $\text{DTr} N$ is finite dimensional.*

Proof. Suppose that M is infinite dimensional. If N is finite dimensional, then f is an epimorphism. By Lemma 3.1, the kernel of f is finite dimensional, and consequently, M is finite dimensional. This contradiction shows that N is infinite dimensional. For proving the second part of the lemma, we may assume that N is indecomposable and not projective. Then $\text{rep}(Q)$ has an almost split sequence

$$\eta : 0 \longrightarrow L \longrightarrow E \xrightarrow{g} N \longrightarrow 0,$$

where $L \in \text{rep}^-(Q)$. Suppose that L is infinite dimensional. By Corollary 1.7, $\text{supp} L$ has a left infinite path. Since $\text{supp} M$ has no left infinite path by Lemma 1.6(2), there exists some $a \in Q_0$ such that $L(a) \neq 0$ but $M(a) = 0$. Let Σ be the successor-closed subquiver of Q generated by a and the vertices in the support of $M \oplus N$. Then Σ is top-finite. By Proposition 2.17(1), restricting η to Σ , we get an almost split sequence

$$\eta_\Sigma : 0 \longrightarrow L_\Sigma \longrightarrow E_\Sigma \xrightarrow{g_\Sigma} N \longrightarrow 0$$

in $\text{rep}(\Sigma)$. By the dual of Lemma 2.15(1), $L_\Sigma \in \text{rep}^-(\Sigma)$, and hence $\text{supp} L_\Sigma$ is socle-finite by Lemma 1.6. On the other hand, since $\text{supp} L_\Sigma$ is a subquiver of the top-finite quiver Σ , it is finite. As a consequence, η_Σ lies in $\text{rep}^+(\Sigma)$ and hence, it is an almost split sequence in $\text{rep}^+(\Sigma)$.

Finally, by Lemma 2.15(2), f lies in $\text{rep}^+(\Sigma)$, and hence it is an irreducible morphism in $\text{rep}^+(\Sigma)$. Thus we have an irreducible morphism $h : L_\Sigma \rightarrow M$. Since L_Σ is finite dimensional while M is infinite dimensional, h is a monomorphism. Since $L_\Sigma(a) = L(a) \neq 0$, we have $M(a) \neq 0$, a contradiction. The proof of the lemma is completed.

3.3. COROLLARY. *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an almost split sequence in $\text{rep}(Q)$ with $L \in \text{rep}^-(Q)$ and $N \in \text{rep}^+(Q)$, and let X be an indecomposable direct summand of M .*

- (1) *If L is infinite dimensional, then X is finitely presented if and only if X is finite dimensional.*
- (2) *If N is infinite dimensional, then X is finitely co-presented if and only if X is finite dimensional.*

Proof. By assumption, there exists an irreducible morphism $f : X \rightarrow N$ in $\text{rep}(Q)$. Suppose that $X \in \text{rep}^+(Q)$. Then f is irreducible in $\text{rep}^+(Q)$. If X is infinite dimensional, then L is finite dimensional by Lemma 3.2. This proves Statement (1). Using the dual of Lemma 3.2, we may prove Statement (2). The proof of the corollary is completed.

Next, we shall relate the irreducibility of a morphism in $\text{rep}^+(Q)$ to its irreducibility in $\text{rep}(Q)$.

3.4. LEMMA. *Let M be an indecomposable representation in $\text{rep}^+(Q)$.*

- (1) *If M is finite dimensional, then $\text{rep}^+(Q)$ has a minimal left almost split morphism $f : M \rightarrow N$, which is also minimal left almost split in $\text{rep}(Q)$.*
- (2) *If $\text{DTr} M$ is finite dimensional, then $\text{rep}^+(Q)$ has a minimal right almost split morphism $g : L \rightarrow M$, which is also minimal right almost split in $\text{rep}(Q)$.*

Proof. (1) Suppose that $M \in \text{rep}^b(Q)$. If M is injective then $M \cong I_x$ for some $x \in Q_0$ by Lemma 1.16(2), and in this case, the projection $p : M \rightarrow M/\text{soc } M$ is minimal left almost split in $\text{rep}^+(Q)$ and in $\text{rep}(Q)$. Otherwise, by Corollary 2.9(2), $\text{rep}(Q)$ has a minimal left almost split morphism $f : M \rightarrow N$ which lies in $\text{rep}^+(Q)$. Thus, f is a minimal left almost split morphism in $\text{rep}^+(Q)$.

(2) Suppose that $\text{DTr } M \in \text{rep}^b(Q)$. If $\text{DTr } M = 0$, then $M \cong P_x$ for some $x \in Q_0$, and in this case, the inclusion $q : \text{rad } M \rightarrow M$ is minimal right almost split in $\text{rep}(Q)$ and in $\text{rep}^+(Q)$. Otherwise, by Corollary 2.9(1), $\text{rep}(Q)$ has a minimal right almost split morphism $g : L \rightarrow M$, which lies in $\text{rep}^+(Q)$. Hence, g is a minimal right almost split morphism in $\text{rep}^+(Q)$. The proof of the lemma is completed.

3.5. COROLLARY. *Let $f : M \rightarrow N$ be a morphism in $\text{rep}^+(Q)$ with M, N indecomposable. Then f is irreducible in $\text{rep}^+(Q)$ if and only if it is irreducible in $\text{rep}(Q)$.*

Proof. Suppose that f is irreducible in $\text{rep}^+(Q)$ with M, N indecomposable. Assume first that $M \in \text{rep}^b(Q)$. By Lemma 3.4(1), $\text{rep}^+(Q)$ has a minimal left almost split morphism $g : M \rightarrow L$, which is minimal left almost split in $\text{rep}(Q)$. If f is irreducible in $\text{rep}^+(Q)$, then $f = ug$ for some retraction $u : L \rightarrow N$, and hence, f is irreducible in $\text{rep}(Q)$; see [6, (2.4)].

Assume now that M is infinite dimensional. By Lemma 3.2, $\text{DTr } N \in \text{rep}^b(Q)$. By Lemma 3.4(2), $\text{rep}^+(Q)$ has a minimal right almost split morphism $h : L \rightarrow N$, which is minimal right almost split in $\text{rep}(Q)$. If f is irreducible in $\text{rep}^+(Q)$, then $f = hv$ for some section $v : N \rightarrow L$, and consequently, f is irreducible in $\text{rep}(Q)$. The proof of the corollary is completed.

The following result is essential for our later investigation, since it allows us to apply some classical techniques used in the finite case.

3.6. PROPOSITION. *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence in $\text{rep}(Q)$, then it is an almost split sequence in $\text{rep}^+(Q)$ if and only if it is an almost split sequence in $\text{rep}(Q)$ with $L \in \text{rep}^b(Q)$.*

Proof. The sufficiency follows from Corollary 2.9(2) and the uniqueness of an almost split sequence. For the necessity, assume that $\eta : 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an almost split sequence in $\text{rep}^+(Q)$. By Lemma 3.1, $L \in \text{rep}^b(Q)$. Since L is not injective, by Corollary 2.9(2), $\text{rep}(Q)$ has an almost split sequence ζ starting with L , which is also an almost split sequence in $\text{rep}^+(Q)$. Then ζ is isomorphic to η . In other words, η is an almost split sequence in $\text{rep}(Q)$. The proof of the proposition is completed.

We are ready to give conditions for $\text{rep}^+(Q)$ to be left or right Auslander-Reiten.

3.7. THEOREM. *If Q is a strongly locally finite quiver, then*

- (1) $\text{rep}^+(Q)$ is left Auslander-Reiten if and only if Q has no right infinite path;
- (2) $\text{rep}^+(Q)$ is right Auslander-Reiten if and only if Q has no left infinite path, or else Q is a left infinite or double infinite path.

Proof. (1) Let Q be a strongly locally finite quiver. Suppose first that Q has a right infinite path p with an initial arrow $x \rightarrow y$. In particular, P_y is infinite dimensional. By Proposition 3.6, $\text{rep}^+(Q)$ admits no almost split sequence starting with P_y .

Suppose that $\text{rep}^+(Q)$ has a minimal left almost split epimorphism $f : P_y \rightarrow L$. By Lemma 2.1(2), P_y is injective in $\text{rep}^+(Q)$. In particular, the inclusion $g : P_y \rightarrow P_x$ is a section, which is absurd. Thus $\text{rep}^+(Q)$ is not left Auslander-Reiten. Conversely, assume that Q contains no right infinite path. Then $\text{rep}^+(Q) = \text{rep}^b(Q)$. Let M be an indecomposable object in $\text{rep}^b(Q)$. If M is not injective, then $\text{rep}^+(Q)$ admits an almost split sequence starting with M by Corollary 2.9(2). Otherwise, by Proposition 1.16(2), $M = I_x$ for some $x \in Q^+$. Thus $M \rightarrow M/\text{soc}M$ is a minimal left almost split epimorphism in $\text{rep}^+(Q)$. That is, $\text{rep}^+(Q)$ is left Auslander-Reiten.

(2) For proving the sufficiency, let $N \in \text{rep}^+(Q)$ be indecomposable. If N is projective, then the inclusion $g : \text{rad}N \rightarrow N$ is a minimal right almost split monomorphism in $\text{rep}^+(Q)$. Otherwise, $\text{rep}^+(Q)$ admits an almost split sequence

$$\eta : 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

where L is an indecomposable non-injective object in $\text{rep}^-(Q)$. If Q contains no left infinite path, then $\text{rep}^-(Q) = \text{rep}^b(Q)$, and hence L is finite dimensional. If Q is a left infinite or double infinite path, then every indecomposable non-injective object in $\text{rep}^-(Q)$ is finite dimensional; see (5.9)(2) below, and hence L is finite dimensional. In any case, by Proposition 3.6, η is an almost split sequence in $\text{rep}^+(Q)$. This shows that $\text{rep}^+(Q)$ is right Auslander-Reiten.

Conversely, assume that $\text{rep}^+(Q)$ is right Auslander-Reiten. By Proposition 3.6, $\text{DTr}X \in \text{rep}^b(Q)$ for any indecomposable non-projective object X in $\text{rep}^+(Q)$. Suppose that Q contains a left infinite path p . Choose arbitrarily a vertex a lying on p . Then Q contains a left infinite path

$$\cdots \longrightarrow a_n \xrightarrow{\alpha_n} a_{n-1} \longrightarrow \cdots \longrightarrow a_1 \xrightarrow{\alpha_1} a_0 = a.$$

We claim that a is the starting point of at most one arrow, while α_1 is the only arrow ending in a . Indeed, assume that $a^+ = \{\alpha_i : a \rightarrow b_i \mid i = 1, \dots, r\}$ with $r > 1$. Then S_a is not projective with a minimal projective resolution

$$0 \longrightarrow \bigoplus_{i=1}^r P_{b_i} \longrightarrow P_a \longrightarrow S_a \longrightarrow 0.$$

By Lemma 2.5, $\text{DTr}S_a$ has a minimal injective co-resolution

$$0 \longrightarrow \text{DTr}S_a \longrightarrow \bigoplus_{i=1}^r I_{b_i} \longrightarrow I_a \longrightarrow 0.$$

For each $n \geq 0$, since $\dim I_{b_i}(a_n) \geq \dim I_a(a_n) > 0$ for all $1 \leq i \leq r$, we get

$$\dim(\text{DTr}S_a)(a_n) = \sum_{i=1}^r \dim I_{b_i}(a_n) - \dim I_a(a_n) \geq \sum_{i=2}^r \dim I_{b_i}(a_n) > 0.$$

Therefore, $\text{DTr}S_a$ is infinite dimensional, a contradiction. Next, assume that there exists an arrow $\beta : b \rightarrow a$ different from α_1 . Similarly, $\text{DTr}S_b$ has a minimal injective co-resolution

$$0 \longrightarrow \text{DTr}S_b \longrightarrow I_a \oplus I \longrightarrow I_b \longrightarrow 0,$$

where $I \in \text{inj}(Q)$. Note, for each $n \geq 1$, that there exists a k -monomorphism $\phi_n : I_b(a_n) \rightarrow I_a(a_n) : \rho \mapsto \beta\rho$. Since $\alpha_1 \neq \beta$, the path $\alpha_1 \cdots \alpha_n$ lies in $I_a(a_n)$ but not in the image of ϕ_n . As a consequence, $\dim I_a(a_n) > \dim I_b(a_n)$. Therefore,

$$\dim(\text{DTr}S_b)(a_n) = \dim I_a(a_n) + \dim I(a_n) - \dim I_b(a_n) > 0,$$

for all $n \geq 1$. In particular, $\text{DTr}S_b$ is infinite dimensional, a contradiction. Our claim is established, from which we infer that Q is a double infinite path if p is contained in a double infinite path, and otherwise, Q is a left infinite path. The proof of the theorem is completed.

We conclude this section with an immediate consequence of Theorem 3.7.

3.8. COROLLARY. *If Q is a strongly locally finite quiver, then $\text{rep}^+(Q)$ is Auslander-Reiten if and only if either Q has no infinite path or Q is a left infinite path.*

4. AUSLANDER-REITEN COMPONENTS

The objective of this section is to describe the Auslander-Reiten components of $\text{rep}^+(Q)$, which has been shown to be a Hom-finite abelian k -category. In contrast to the finite case, we shall see that many new phenomena occur.

First of all, we recall the notion of a section of a translation quiver, which is essential in describing the Auslander-Reiten components. Let Γ be a connected valued or non-valued translation quiver with translation τ ; see, for example, [22, 41]. A connected convex subquiver Δ of Γ is called a *section* if it contains no oriented cycle and meets each τ -orbit exactly once; see [32, (2.1)]. Now, we say that a section Δ of Γ is *right-most* or *left-most* if the vertices in Γ are all of the form $\tau^n x$ or all of the form $\tau^{-n} x$ with $n \in \mathbb{N}$ and $x \in \Delta_0$, respectively.

4.1. LEMMA. *Let (Γ, τ) be a connected translation quiver with no oriented cycle, and let Δ be a full subquiver of Γ meeting any given τ -orbit at most once.*

- (1) *If Δ is successor-closed in Γ and has the following property: for each arrow $x \rightarrow \tau^n y$ in Γ with $n \geq 0$, $y \in \Delta$ implies $x \in \Delta$ or $\tau^{-n} x \in \Gamma$, then it is a right-most section of Γ .*
- (2) *If Δ is predecessor-closed in Γ and has the following property: for each arrow $\tau^{-n} x \rightarrow y$ in Γ with $n \geq 0$, $x \in \Delta$ implies $y \in \Delta$ or $\tau y \in \Gamma$, then it is a left-most section of Γ .*

Proof. We shall prove only the first statement. Assume that Δ satisfies the condition stated in (1). Let Σ be a connected component of Δ . Then Σ contains no oriented cycle and meets any τ -orbit in Γ at most once. Moreover, since Δ is successor-closed in Γ , so is Σ . In particular, Σ is convex in Γ and the vertices in the τ -orbit of some vertex z in Σ are all of the form $\tau^r z$ with $r \geq 0$. We claim that every vertex a in Γ lies in the τ -orbit of some vertex in Σ . Since Γ is connected, we may assume that Γ contains an edge $a - b$, where b lies in the τ -orbit of some $x \in \Sigma_0$. Then $b = \tau^n x$ for some $n \geq 0$. If $\tau^{-n-1} a \in \Gamma$, then either $x \rightarrow \tau^{-n-1} a$ or $x \rightarrow \tau^{-n} a$ is an arrow in Γ . Since Σ is successor-closed in Γ , we have $\tau^{-n} a \in \Sigma$ or $\tau^{-n-1} a \in \Sigma$. Suppose now that $\tau^{-n-1} a \notin \Gamma$. Then there exists some $0 \leq m \leq n$ such that $\tau^{-m} a \in \Gamma$ while $\tau^{-m-1} a \notin \Gamma$. This yields an arrow $\tau^{n-m} x \rightarrow \tau^{-m} a$ or $\tau^{-m} a \rightarrow \tau^{n-m} x$ in Γ . If $n = m$, since $\tau^{-m-1} a \notin \Gamma$, it follows from the condition stated in (1) that $\tau^{-m} a \in \Delta$, and hence $\tau^{-m} a \in \Sigma$. If $m < n$, then either $\tau^{-m} a \rightarrow \tau^{n-m-1} x$ or $\tau^{-m} a \rightarrow \tau^{n-m} x$ is an arrow in Γ . By the property of Δ stated in (1), we have $\tau^{-m} a \in \Delta$. Since Δ is successor-closed, we get $\tau^{n-m-1} x \in \Delta$. Since Δ meets any τ -orbit at most once, we see that $m = n - 1$ and $\tau^{-m} a \rightarrow x$ is an arrow in Γ . Thus $\tau^{-m} a \in \Sigma$. This establishes our claim. As a consequence, Σ is a right-most section of Γ . Finally, since Δ meets any τ -orbit at most once, we have $\Delta = \Sigma$. The proof of the lemma is completed.

Let (Γ, τ) be a connected valued or non-valued translation quiver, and let x be a vertex in Γ . One says that x is *projective* or *injective* if τx or $\tau^{-1} x$ is not defined

in Γ , respectively. Moreover, we say that x is *left stable* if $\tau^n x$ is defined for all $n \in \mathbb{N}$; *right stable* if $\tau^{-n} x$ is defined for all $n \in \mathbb{N}$; and *stable* if it is both left and right stable. Furthermore, Γ is called *left stable*, *right stable*, or *stable* if its vertices are all left stable, all right stable, or all stable, respectively.

Given a connected quiver Δ with no oriented cycle, one constructs a stable translation quiver $\mathbb{Z}\Delta$; see, for example, [32, Section 2]. We denote by $\mathbb{N}\Delta$ the full translation subquiver of $\mathbb{Z}\Delta$ generated by the vertices (n, x) with $n \geq 0$ and $x \in \Delta_0$, and by $\mathbb{N}^-\Delta$ the one generated by the vertices (n, x) with $n \leq 0$ and $x \in \Delta_0$. It is evident that $\mathbb{N}\Delta$ is right stable with a left-most section generated by the vertices $(0, x)$ with $x \in \Delta_0$, while $\mathbb{N}^-\Delta$ is left stable with a right-most section generated by the vertices $(0, x)$ with $x \in \Delta_0$. Assume that Δ is a section of Γ . Then Γ is isomorphic to the full translation subquiver of $\mathbb{Z}\Delta$ generated by the vertices $(-n, x)$, where $n \in \mathbb{Z}$ and $x \in \Delta_0$ such that $\tau^n x \in \Gamma$; see [32, (2.3)]. In particular, if Δ is left-most or right-most, then Γ embeds in $\mathbb{N}\Delta$ or $\mathbb{N}^-\Delta$, respectively.

Let \mathcal{A} be a Hom-finite Krull-Schmidt additive k -category. For indecomposable objects $X, Y \in \mathcal{A}$, we write

$$\text{irr}(X, Y) = \text{rad}(X, Y) / \text{rad}^2(X, Y)$$

and denote its dimensions over $\text{End}(X) / \text{rad}(X, X)$ and $\text{End}(Y) / \text{rad}(Y, Y)$ by d'_{XY} and d_{XY} , respectively. It is then well known that \mathcal{A} has an irreducible morphism $f : X \rightarrow Y$ if and only if $d_{XY} > 0$, and in this case, d'_{XY} and d_{XY} are maximal such that \mathcal{A} has irreducible morphisms $g : X^{d'_{XY}} \rightarrow Y$ and $h : X \rightarrow Y^{d_{XY}}$, respectively, where M^n denotes the direct sum of n copies of M ; see [9, (3.4)]. The *Auslander-Reiten quiver* $\Gamma_{\mathcal{A}}$ of \mathcal{A} is a valued translation quiver defined as follows: the vertex set is a complete set of representatives of isomorphism classes of the indecomposable objects in \mathcal{A} ; for vertices X, Y , there exists a unique valued arrow $X \rightarrow Y$ with valuation (d_{XY}, d'_{XY}) if and only if $d_{XY} > 0$; and the translation τ is defined so that $\tau Z = X$ if and only if \mathcal{A} has an almost split sequence $X \twoheadrightarrow Y \twoheadrightarrow Z$; see [33, (2.1)]. A valuation (d_{XY}, d'_{XY}) is called *symmetric* if $d_{XY} = d'_{XY}$, and *trivial* if $d_{XY} = d'_{XY} = 1$. For some technical reasons; see (4.3), we replace each symmetrically valued arrow $X \rightarrow Y$ by d_{XY} unvalued arrows from X to Y . In this way, $\Gamma_{\mathcal{A}}$ becomes a partially valued translation quiver with possible multiple arrows in which all valuations are non-symmetric. The connected components of $\Gamma_{\mathcal{A}}$ are called the *Auslander-Reiten components* of \mathcal{A} .

Now, we specialize to the Auslander-Reiten quiver $\Gamma_{\text{rep}^+(Q)}$ of $\text{rep}^+(Q)$. We choose its vertex set in such a way that it contains the S_x with $x \in Q_0$, the P_x with $x \in Q_0$, and the I_x with $x \in Q^+$. Recall that Q^+ denotes the full subquiver of Q generated by the vertices x such that I_x is finite dimensional. We shall see that the arrows in $\Gamma_{\text{rep}^+(Q)}$ are all symmetrically valued; see (4.15) below, and hence $\Gamma_{\text{rep}^+(Q)}$ is an unvalued translation quiver with multiple arrows in the sense of [41, page 47]. For convenience, we say that a representation M in $\Gamma_{\text{rep}^+(Q)}$ is *pseudo-projective* if $\text{DTr} M$ is infinite dimensional, or equivalently, $\text{DTr} M \notin \text{rep}^+(Q)$. The following observation, which follows from Proposition 3.6 and Corollary 2.9, clarifies the relation between the Auslander-Reiten translations τ and τ^- for $\Gamma_{\text{rep}^+(Q)}$ and the Auslander-Reiten translations DTr and TrD for $\text{rep}(Q)$.

4.2. LEMMA. *If M is a representation lying in $\Gamma_{\text{rep}^+(Q)}$, then*

- (1) τM is defined in $\Gamma_{\text{rep}^+(Q)}$ if and only if M is neither projective nor pseudo-projective, and in this case, $\tau M \cong \text{DTr} M$, which is of positive finite dimension;
- (2) $\tau^- M$ is defined in $\Gamma_{\text{rep}^+(Q)}$ if and only if M is finite dimensional and not injective, and in this case, $\tau^- M \cong \text{Tr} D M$;
- (3) M is left stable or right stable in $\Gamma_{\text{rep}^+(Q)}$ if and only if M is DTr-stable or TrD-stable in $\text{rep}(Q)$, respectively.

REMARK. In other words, M is a projective vertex in $\Gamma_{\text{rep}^+(Q)}$ if and only if M is a projective or pseudo-projective representation in $\text{rep}^+(Q)$; and M is an injective vertex in $\Gamma_{\text{rep}^+(Q)}$ if and only if M is an injective or infinite dimensional representation in $\text{rep}^+(Q)$.

4.3. LEMMA. *Let P_Q be the full subquiver of $\Gamma_{\text{rep}^+(Q)}$ generated by the P_x with $x \in Q_0$, and let I_Q be the one generated by the I_x with $x \in Q^+$.*

- (1) *The subquiver P_Q is predecessor-closed in $\Gamma_{\text{rep}^+(Q)}$ and isomorphic to Q^{op} .*
- (2) *The subquiver I_Q is successor-closed in $\Gamma_{\text{rep}^+(Q)}$ and isomorphic to $(Q^+)^{\text{op}}$.*

Proof. We prove only the first statement, since the second one follows dually. For $x, y \in Q_0$, denote by n_{xy} the number of arrows in Q from x to y . By Proposition 1.3, $\text{End}(P_x) \cong \text{End}(P_y) \cong k$ and $\text{irr}(P_y, P_x)$ has k -dimension n_{xy} . Thus $\Gamma_{\text{rep}^+(Q)}$ contains a valued arrow $P_y \rightarrow P_x$ if and only if $n_{xy} > 0$, and in this case, the valuation is (n_{xy}, n_{xy}) . By definition, the symmetrically valued arrow $P_y \rightarrow P_x$ is replaced by n_{xy} unvalued arrows from P_y to P_x . Thus, $P_Q \cong Q^{\text{op}}$. Moreover, if $M \rightarrow P_x$ with $x \in Q$ is an arrow in $\Gamma_{\text{rep}^+(Q)}$, then M is a direct summand of $\text{rad} P_x$, and hence $M = P_y$ for some $y \in Q$. Thus P_Q is predecessor-closed in $\Gamma_{\text{rep}^+(Q)}$. The proof of the lemma is completed.

The following result is well known in the finite case.

4.4. LEMMA. *If Q is connected, then $\Gamma_{\text{rep}^+(Q)}$ contains an oriented cycle if and only if Q is finite of Euclidean type.*

Proof. We only need to consider the case where Q is connected and infinite. Suppose that $\Gamma_{\text{rep}^+(Q)}$ contains an oriented cycle

$$\eta : \quad M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_n = M_1.$$

Since Q has no oriented cycle, it follows from Lemma 4.3 that none of the M_i is projective. If some of the M_i is infinite dimensional, then the M_i are all infinite dimensional and the $\text{DTr} M_i$ are of finite positive dimension by Lemma 3.2. Thus $\Gamma_{\text{rep}^+(Q)}$ has an oriented cycle

$$\tau M_1 \longrightarrow \tau M_2 \longrightarrow \cdots \longrightarrow \tau M_n = \tau M_1,$$

which contains only finite dimensional representations. Thus, we may assume that the M_i with $1 \leq i \leq n$ are all finite dimensional. In particular, $\text{supp}(M_1 \oplus \cdots \oplus M_n)$ is contained in a finite connected full subquiver Σ of Q . Observing that η is also an oriented cycle in $\Gamma_{\text{rep}(\Sigma)}$, we see that Σ is of Euclidean type. Since Q is connected and infinite, Σ is contained in a connected finite full subquiver Δ of Q which is of wild type. Again, η is an oriented cycle in $\Gamma_{\text{rep}(\Delta)}$, a contradiction. The proof of the lemma is completed.

The preceding lemma yields the following important consequence.

4.5. LEMMA. *Let Γ be a connected component of $\Gamma_{\text{rep}^+(Q)}$.*

- (1) *If Γ contains infinite dimensional representations, then such representations generate a right-most section of Γ .*
- (2) *If Γ contains pseudo-projective representations, then such representations generate a left-most section of Γ .*

Proof. (1) By Lemma 4.4, Γ has no oriented cycle. Assume that the full subquiver Δ of Γ generated by the infinite dimensional representations is non-empty. By Proposition 3.6, Δ meets any τ -orbit in Γ at most once, and by Lemma 3.2, Δ is successor-closed in Γ . Let $M \rightarrow \tau^n N$ be an arrow in Γ , where $n \geq 0$, $N \in \Delta$, and $M \notin \Delta$. Then M is finite dimensional. If M is injective then, by Proposition 1.16, $M = I_x$ for some $x \in Q^+$. By Lemma 4.3, $n = 0$ and $N = I_y$ for some $y \in Q^+$, contrary to that $N \in \Delta$. Thus M is not injective, and by Lemma 4.2(2), $\tau^- M \in \Gamma$. It follows then from Lemma 4.1(1) that Δ is a right-most section of Γ .

(2) Assume that the full subquiver Σ of Γ generated by the pseudo-projective representations is non-empty. Clearly, Σ meets any τ -orbit in Γ at most once. Fix an arrow $M \rightarrow N$ in Γ . Suppose first that $N \in \Sigma$. Then $\text{rep}(Q)$ admits an almost split sequence $0 \rightarrow \text{DTr} N \rightarrow E \rightarrow N \rightarrow 0$, where $\text{DTr} N \notin \text{rep}^+(Q)$. By Corollary 3.5, an irreducible morphism $f : M \rightarrow N$ in $\text{rep}^+(Q)$ is irreducible in $\text{rep}(Q)$. Thus there exists an irreducible morphism $g : \text{DTr} N \rightarrow M$ in $\text{rep}(Q)$. If $M \notin \Sigma$, by Lemma 3.4(2), $\text{rep}^+(Q)$ has a minimal right almost split morphism $h : L \rightarrow M$ which is minimal right almost split in $\text{rep}(Q)$. Therefore, $\text{DTr} N$ is a direct summand of L , and hence $\text{DTr} N \in \text{rep}^+(Q)$, a contradiction. Therefore, $M \in \Sigma$. In particular, Σ is predecessor-closed in Γ . Suppose next that $M = \tau^{-n} X$ with $n \geq 0$ and $X \in \Sigma$. If N is projective, then M is projective, and hence $n = 0$ and X is projective, which contradicts that X is pseudo-projective. Thus N is not projective, and hence either N is pseudo-projective or τN is defined in Γ , that is, either $N \in \Sigma$ or $\tau N \in \Gamma$. By Lemma 4.1(2), Σ is a left-most section of Γ . The proof of the lemma is completed.

We are ready to describe the connected components of $\Gamma_{\text{rep}^+(Q)}$. Such a connected component is called *preprojective* if it contains some of the P_x with $x \in Q_0$. In case Q is connected, by Lemma 4.3(1), $\Gamma_{\text{rep}^+(Q)}$ has a unique preprojective component which we denote by \mathcal{P}_Q .

4.6. THEOREM. *Let Q be an infinite connected strongly locally finite quiver. The preprojective component \mathcal{P}_Q of $\Gamma_{\text{rep}^+(Q)}$ has a left-most section generated by the P_x with $x \in Q_0$, and consequently, \mathcal{P}_Q embeds in NQ^{op} . Furthermore,*

- (1) *If Q has no right infinite path, then \mathcal{P}_Q is right stable of shape NQ^{op} .*
- (2) *If Q has right infinite paths, then \mathcal{P}_Q has a right-most section, and consequently, \mathcal{P}_Q contains only finite τ -orbits.*

Proof. By Lemma 4.3(1), the full subquiver \mathcal{P}_Q of \mathcal{P}_Q generated by the P_x with $x \in Q_0$ is predecessor-closed in \mathcal{P}_Q and is isomorphic to Q^{op} . Clearly, \mathcal{P}_Q meets at most once any given τ -orbit in Γ . Let $\tau^{-n} M \rightarrow N$ be an arrow in \mathcal{P}_Q with $n \geq 0$ and $M \in \mathcal{P}_Q$. Then $\tau^{-n} M$ is not pseudo-projective. By Lemma 3.4(2), $\text{rep}^+(Q)$ has a minimal right almost split morphism $f : L \rightarrow \tau^{-n} M$, which is minimal right

almost split in $\text{rep}(Q)$. If $N \notin \mathcal{P}_Q$, then $\text{rep}(Q)$ has an almost split sequence

$$0 \longrightarrow \text{DTr}N \longrightarrow E \longrightarrow N \longrightarrow 0.$$

In view of Corollary 3.5, we see that $\text{rep}(Q)$ admits an irreducible morphism $g : \tau^{-n}M \rightarrow N$, and hence an irreducible morphism $h : \text{DTr}N \rightarrow \tau^{-n}M$. As a consequence, $\text{DTr}N$ is a direct summand of L . In particular, $\text{DTr}N \in \text{rep}^+(Q)$, and therefore, $\tau N \in \mathcal{P}_Q$. By Lemma 4.1(2), \mathcal{P}_Q is a left-most section of \mathcal{P}_Q . In particular, \mathcal{P}_Q embeds in $\mathbb{N}Q^{\text{op}}$; see [32, (2.3)].

Furthermore, if Q has no right infinite path, then $\text{rep}^+(Q) = \text{rep}^b(Q)$. In particular, \mathcal{P}_Q contains only finite dimensional representation. Containing no injective representation by Proposition 2.14(1) and Corollary 1.16(2), \mathcal{P}_Q is right stable by Lemma 4.2(2). As a consequence, $\mathcal{P}_Q \cong \mathbb{N}Q^{\text{op}}$. Otherwise, some of the P_x are infinite dimensional, and hence \mathcal{P}_Q has a right-most section by Lemma 4.5. Now, since \mathcal{P}_Q has a left-most section and a right-most one, every τ -orbit in \mathcal{P}_Q is finite. The proof of the theorem is completed.

REMARK. In case Q has right infinite paths, we can describe \mathcal{P}_Q more explicitly in the following way. Consider the right stable translation quiver $\mathbb{N}Q^{\text{op}}$. We first define $f(0, x) = \dim_k P_x \in \mathbb{N} \cup \{\infty\}$ for $x \in Q_0$, and then extend this in a unique way to an additive function

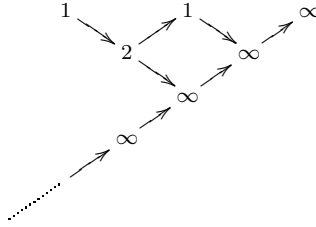
$$f : \mathbb{N}Q^{\text{op}} \rightarrow \mathbb{N} \cup \{\infty\}$$

such that $f(v) = \infty$ whenever v is a successor of some u for which $f(u) = \infty$. Then \mathcal{P}_Q is isomorphic to the full translation subquiver of $\mathbb{N}Q^{\text{op}}$ generated by the vertices (n, x) with $n \in \mathbb{N}$ and $x \in Q_0$ such that $n = 0$, or otherwise, $f(n-1, x) < \infty$.

EXAMPLE. If Q is the infinite quiver

$$\circ \longleftarrow \circ \longleftarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \dots,$$

then the preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is of the following shape :

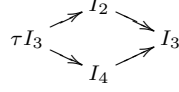


where each vertex is labeled with the dimension of the corresponding representation.

Next, we shall describe the connected components of $\Gamma_{\text{rep}^+(Q)}$ containing some of the I_x with $x \in Q^+$, called the *preinjective components*. To do so, for each $x \in Q^+$, we denote by Q_x^+ the connected component of Q^+ containing x .

4.7. THEOREM. *Let Q be an infinite connected strongly locally finite quiver. If Γ is a preinjective component of $\Gamma_{\text{rep}^+(Q)}$ containing I_x for some $x \in Q^+$, then it has a right-most section generated by the I_y with $y \in Q_x^+$, and consequently, it contains only finite dimensional representations and embeds in $\mathbb{N}^-(Q_x^+)^{\text{op}}$. Furthermore,*

then $\Gamma_{\text{rep}^+(Q)}$ has a trivial preinjective component $\{I_0\}$ and another preinjective component of the following shape :



A representation lying in $\Gamma_{\text{rep}^+(Q)}$ is called *preprojective* or *preinjective* if it lies in a preprojective component or in a preinjective component, respectively. Before going further, we shall study some properties of these representations.

4.8. LEMMA. *Let M be a representation lying in $\Gamma_{\text{rep}^+(Q)}$.*

- (1) *If M is preprojective, then it has only finitely many non-projective predecessors in the preprojective component.*
- (2) *If M is preinjective, then it has only finitely many successors in its preinjective component.*

Proof. We may assume that Q is connected. Suppose first that M lies in the preprojective component \mathcal{P}_Q . By Theorem 4.6 and Lemma 4.3, \mathcal{P}_Q has a left-most section P_Q , which is generated by the P_x with $x \in Q_0$ and isomorphic to Q^{op} . Suppose that M has infinitely many non-projective predecessors in \mathcal{P}_Q . By König's lemma, \mathcal{P}_Q has a left infinite path of non-projective representations as follows :

$$\cdots \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 = M .$$

Since P_Q is a left-most section, we can write $M_i = \tau^{-r_i} P_{x_i}$, where $x_i \in Q_0$ and $r_i \in \mathbb{N}$ such that $r_i \geq r_{i+1} > 0$ for all $i \geq 0$. Thus, we may assume that $r_i = r_0$ for all $i \geq 1$. This implies that P_Q has a left infinite path ending in P_{x_0} , which in turn implies that Q has a right infinite path starting in x_0 . In particular, P_{x_0} is infinite dimensional. By Lemma 4.2(2), $\tau^- P_{x_0}$ is not defined, which is absurd since $r_0 > 0$. This proves Statement (1). Since the preinjective components of $\Gamma_{\text{rep}^+(Q)}$ contain only finite dimensional representations, we can prove Statement (2) in a dual manner. The proof of the lemma is completed.

The following two statements are well known in the finite case.

4.9. LEMMA. *Let $f : M \rightarrow N$ be a non-zero non-invertible morphism in $\text{rep}^+(Q)$ with $M, N \in \Gamma_{\text{rep}^+(Q)}$. If M is preinjective or N is preprojective, then $\Gamma_{\text{rep}^+(Q)}$ contains a non-trivial path from M to N .*

Proof. We consider only the case where N lies in the preprojective component \mathcal{P}_Q of $\Gamma_{\text{rep}^+(Q)}$, since the other case can be treated in a dual manner. Let P_Q be the left-most section in \mathcal{P}_Q . Suppose first that $N = P_y$ for some $y \in Q_0$. Since f is non-zero and $\text{rep}^+(Q)$ is hereditary, $M = P_x$ for some $x \in Q_0$. Since f is non-invertible, we deduce from Proposition 1.3 that x is a proper successor of y in Q . Hence P_y is a proper successor of P_x in P_Q . Suppose now that N is not projective while M is not a proper predecessor of N in \mathcal{P}_Q . Since every representation in \mathcal{P}_Q is the co-domain of a minimal right almost split morphism in $\text{rep}^+(Q)$, using induction, we get an infinite path

$$\cdots \longrightarrow N_i \longrightarrow N_{i-1} \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 = N$$

in \mathcal{P}_Q and non-zero non-invertible morphisms $f_i : M \rightarrow N_i$ for $i \geq 0$. By lemma 4.8, there exists a positive integer n such that N_n is projective. Hence, M is a proper

predecessor of N_n by our previous consideration, and hence a proper predecessor of N in \mathcal{P}_Q , a contradiction. The proof of the lemma is completed.

4.10. PROPOSITION. *Let M be an indecomposable representation lying in $\Gamma_{\text{rep}^+(Q)}$.*

- (1) *If M is preprojective, then $\text{Hom}(L, M) = 0$ for all but finitely many non-projective representations L in $\Gamma_{\text{rep}^+(Q)}$.*
- (2) *If M is preinjective, then $\text{Hom}(M, L) = 0$ for all but finitely many representations L in $\Gamma_{\text{rep}^+(Q)}$.*
- (3) *If M is preprojective or preinjective, then $\text{Ext}^1(M, M) = 0$.*

Proof. The first two statements follow immediately from Lemmas 4.8 and 4.9. Suppose that $\text{rep}^+(Q)$ admits a non-trivial extension $0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$. Then, M is neither projective nor injective. Assume that M lies in the preprojective component \mathcal{P}_Q . If M is pseudo-projective then, by Lemma 4.5(2), \mathcal{P}_Q has a left-most section generated by its pseudo-projective representations, which coincides with the left-most section generated by the P_x with $x \in Q_0$, a contradiction. Hence τM is defined in \mathcal{P}_Q . This yields a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & \tau M & \longrightarrow & N & \longrightarrow & M \longrightarrow 0 \end{array}$$

in $\text{rep}^+(Q)$, where the lower row is an almost split sequence. By Lemma 4.9, $f = 0$, and hence the lower row splits, a contradiction. Suppose next that M lies in a preinjective component \mathcal{I} . Observing that τ^-M is defined in \mathcal{I} , we get a contradiction by a dual argument. The proof of the proposition is completed.

The rest of this section is mainly devoted to describing the *regular components* of $\Gamma_{\text{rep}^+(Q)}$, that is, the connected components which contain none of the P_x and the I_x with $x \in Q_0$.

4.11. LEMMA. *If $f : M \rightarrow N$ is a morphism in $\text{rep}^b(Q)$, then it is irreducible in $\text{rep}^b(Q)$ if and only if it is irreducible in $\text{rep}(Q)$.*

Proof. We only need to prove the necessity. Let $f : M \rightarrow N$ be an irreducible morphism in $\text{rep}^b(Q)$. Suppose that $f = hg$, where $g : M \rightarrow L$ and $h : L \rightarrow N$ are morphisms in $\text{rep}(Q)$. Let Σ be a finite full subquiver of Q , containing the vertices in $\text{supp}(M \oplus N)$ as well as their immediate predecessors and immediate successors in Q . This yields a factorization $f = h_\Sigma g_\Sigma$ in $\text{rep}^b(Q)$. Therefore, g_Σ is a section or h_Σ is a retraction. In the first case, g is a section since Σ contains the immediate predecessors of the vertices in $\text{supp}M$. In the second case, h is a retraction since Σ contains the immediate successors of the vertices in $\text{supp}N$. The proof of the lemma is completed.

The next result reveals the relation between some regular components of $\Gamma_{\text{rep}^+(Q)}$ and those of $\Gamma_{\text{rep}^-(Q)}$.

4.12. PROPOSITION. *Let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$.*

- (1) *If Γ has infinite dimensional but no pseudo-projective representations, then the full translation subquiver of Γ obtained by deleting the infinite dimensional representations is a left stable regular component of $\Gamma_{\text{rep}^-(Q)}$.*

- (2) If Γ has pseudo-projective but no infinite dimensional representations, then Γ is the full translation subquiver of a right stable regular component of $\Gamma_{\text{rep}^-(Q)}$ obtained by deleting the non-empty set of infinite dimensional representations.

Proof. (1) Suppose that Γ contains infinite dimensional but no pseudo-projective representations. By Proposition 4.5(1), the infinite dimensional representations in Γ generate a right-most section Δ . Let Γ' be the full translation subquiver of Γ generated by the representations not in Δ . Containing no projective or pseudo-projective representation, Γ' is left stable. Being finite dimensional, the representations in Γ' may be assumed to all lie in $\Gamma_{\text{rep}^-(Q)}$. Fix representations M, N in Γ' . It follows from Lemma 4.11 that a morphism $f : M \rightarrow N^r$ with $r > 0$ is irreducible in $\text{rep}^+(Q)$ if and only if it is irreducible in $\text{rep}^-(Q)$ and a morphism $g : M^s \rightarrow N$ with $s > 0$ is irreducible in $\text{rep}^+(Q)$ if and only if it is irreducible in $\text{rep}^-(Q)$. Therefore, $M \rightarrow N$ is a valued arrow with valuation (d, d') in Γ' if and only if $M \rightarrow N$ is a valued arrow with valuation (d, d') in $\Gamma_{\text{rep}^-(Q)}$. In particular, Γ' is a full valued subquiver of some connected component \mathcal{C} of $\Gamma_{\text{rep}^-(Q)}$. Next, since M is neither projective nor pseudo-projective, $\text{rep}^+(Q)$ has an almost split sequence $\eta : 0 \rightarrow \tau M \rightarrow E \rightarrow M \rightarrow 0$, where τM is finite dimensional. By Lemma 4.11, η is also an almost split sequence in $\text{rep}^-(Q)$. This shows that Γ' is a predecessor-closed valued translation subquiver of \mathcal{C} . Next, let $M \rightarrow X$ be an arrow in \mathcal{C} with an irreducible morphism $u : M \rightarrow X$ in $\text{rep}^-(Q)$. By the dual of Corollary 3.5, h is irreducible in $\text{rep}(Q)$. On the other hand, since M is finite dimensional and not injective, by Corollary 2.9(2), $\text{rep}^+(Q)$ has an almost split sequence $0 \rightarrow M \rightarrow E \rightarrow \tau^- M \rightarrow 0$, which is also almost split in $\text{rep}(Q)$. Then X is a direct summand of E . If $\tau^- M \in \Gamma'$, then E is finite dimensional and so is X . If $\tau^- M \in \Delta$ then, by Corollary 3.3(2), X is finite dimensional. In any case, $M \rightarrow X$ is an arrow in Γ' . This shows that Γ' is successor-closed in \mathcal{C} , and hence, $\Gamma' = \mathcal{C}$. That is, Γ' is a regular component of $\Gamma_{\text{rep}^-(Q)}$ which is left stable.

(2) Suppose that Γ contains pseudo-projective but no infinite dimensional representations. Since the representations in Γ are finite dimensional and non-injective, Γ is right stable. Using an argument dual to the above one, we see that Γ is a successor-closed valued translation subquiver of a connected component \mathcal{C} of $\Gamma_{\text{rep}^-(Q)}$. Since Γ is right stable, \mathcal{C} has no right-most section. In particular, by the dual of Theorem 4.6 and the dual of Lemma 4.5(2), \mathcal{C} contains no representation M which is injective or pseudo-injective, where M is *pseudo-injective* if $\text{Tr}DM$ is infinite dimensional. Now, fix a pseudo-projective representation N in Γ . Since it is finite dimensional and non-projective, by Corollary 2.9(1), $\text{rep}(Q)$ has an almost split sequence $0 \rightarrow \text{DTr}N \rightarrow E \rightarrow N \rightarrow 0$ with $\text{DTr}N$ of infinite dimension, which is also an almost split sequence in $\text{rep}^-(Q)$. In particular, $\text{DTr}N$ is an infinite dimensional representation in \mathcal{C} . By the dual of Theorem 4.7, \mathcal{C} is not a preprojective component, and hence a regular component of $\Gamma_{\text{rep}^-(Q)}$. By the dual of Statement (1), the full translation subquiver \mathcal{C}' of \mathcal{C} obtained by deleting the infinite dimensional representations is a connected component of $\Gamma_{\text{rep}^+(Q)}$. Since Γ is a connected component of $\Gamma_{\text{rep}^+(Q)}$ which is contained in \mathcal{C}' , we see that Γ and \mathcal{C}' coincide. The proof of the proposition is completed.

We shall also need the following result to deal with the regular components containing infinite dimensional representations.

4.13. LEMMA. *Let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$, containing an infinite dimensional representation M .*

- (1) *If N, L are representations in Γ , then $\text{rep}(Q)$ admits no chain of irreducible monomorphisms $\text{DTr} L \xrightarrow{f} N \xrightarrow{g} L$.*
- (2) *If $M \rightarrow N$ is an arrow in Γ , then $\tau N \in \Gamma$ with $\dim_k \text{DTr} M > \dim_k \tau N$, and any irreducible morphism $f : M \rightarrow N$ in $\text{rep}^+(Q)$ is an epimorphism.*
- (3) *If $M \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n-1} \rightarrow M_n$ is a path in Γ , then $\tau^j M_i \in \Gamma$ for all $i = 1, \dots, n; j = 0, \dots, i$.*
- (4) *If $M \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$ is a path in Γ and $f : M_3 \rightarrow N$ is an irreducible morphism in $\text{rep}^+(Q)$, then N is indecomposable.*
- (5) *If M is not pseudo-projective, then $\text{rep}^+(Q)$ has a minimal right almost split morphism $f : N_1 \oplus N_2 \rightarrow M$, where N_1 is indecomposable of infinite dimension and N_2 is of finite dimension.*

Proof. Let X be a representation lying in Γ . Write $d(X) = \dim_k X \in \mathbb{N} \cup \{\infty\}$. Since Γ is regular, $\text{DTr} X$ is an indecomposable representation in $\text{rep}^-(Q)$. Thus, τX is defined in Γ if and only if X is not pseudo-projective.

(1) Suppose that $\text{rep}(Q)$ admits irreducible monomorphisms $\text{DTr} L \xrightarrow{f} N \xrightarrow{g} L$ where N, L in Γ . Since $\text{DTr} L$ is a finitely co-presented sub-representation of N , making use of Lemma 1.6(2) and Corollary 1.7, we see that $\text{DTr} L$ is finite dimensional, and hence τL is defined in Γ . Now $\text{rep}(Q)$ has an irreducible morphism $g_1 : \text{DTr} N \rightarrow \text{DTr} L$, which is a monomorphism by Corollary 2.7. In particular, $\text{DTr} N$ is finite dimensional, and hence τN is defined in Γ . Applying the same argument to $\tau N \xrightarrow{g_1} \tau L \xrightarrow{f} N$, we get an irreducible monomorphism $f_1 : \text{DTr}^2 L \rightarrow \tau N$ in $\text{rep}(Q)$. Repeating this process, we see that $\tau^i N$ and $\tau^i L$ are defined in Γ for all $i \geq 0$, and $\text{rep}(Q)$ admits an infinite chain of irreducible monomorphisms

$$\cdots \rightarrow \tau^3 L \xrightarrow{f_2} \tau^2 N \xrightarrow{g_2} \tau^2 L \xrightarrow{f_1} \tau N \xrightarrow{g_1} \tau L \xrightarrow{f} N \xrightarrow{g} L,$$

which is absurd since $d(\tau L) < \infty$.

(2) Let $M \rightarrow N$ be an arrow in Γ . By Lemma 3.2, $\text{DTr} N$ is finite dimensional, and hence $\tau N \in \Gamma$, and $\text{rep}(Q)$ has an irreducible monomorphism $g : \tau N \rightarrow M$. Since M is not projective, $\text{rep}(Q)$ has an irreducible morphism $h : \text{DTr} M \rightarrow \tau N$, which is an epimorphism by Statement (1). Hence $\dim_k \text{DTr} M > d(\tau N)$. By Corollary 2.7, every irreducible morphism $f : M \rightarrow N$ in $\text{rep}^+(Q)$ is an epimorphism.

(3) Let $M \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n-1} \rightarrow M_n$ be a path in Γ . By Lemma 3.2, M_i is infinite dimensional while τM_i is defined of finite dimensional, for every $1 \leq i \leq n$. Thus, $\text{rep}^+(Q)$ has irreducible monomorphisms $f_{i,i-1} : \tau M_i \rightarrow M_{i-1}$, $i = 1, \dots, n$, where $M_0 = M$. Let i with $0 < i \leq n$ be such that $\text{rep}^+(Q)$ has a chain of monomorphisms

$$\tau^{n-i} M_n \xrightarrow{f_{n,i}} \tau^{n-i-1} M_{n-1} \rightarrow \cdots \rightarrow \tau M_{i+1} \xrightarrow{f_{i+1,i}} M_i.$$

By Corollary 2.7, $\text{rep}^-(Q)$ has a chain of monomorphisms

$$\text{DTr}^{n-i+1} M_n \xrightarrow{f_{n,i-1}} \text{DTr}^{n-i} M_{n-1} \rightarrow \cdots \rightarrow \text{DTr}^2 M_{i+1} \xrightarrow{f_{i+1,i-1}} \tau M_i \xrightarrow{f_{i,i-1}} M_{i-1}.$$

Since τM_i is finite dimensional, the $\text{DTr}^{j-i+1} M_j$ with $i \leq j \leq n$ are all finite dimensional. That is, $\tau^{j-i+1} M_j$ is defined in Γ , for all $i \leq j \leq n$.

(4) Let $M \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$ be a path in Γ , and let $f : M_3 \rightarrow N$ be an irreducible morphism in $\text{rep}^+(Q)$. Write $N = \bigoplus_{i=1}^n N_i$ with the N_i indecomposable. By Statement (3), $\tau^j M_i$ is defined of finite dimension for $1 \leq i \leq 3$ and $1 \leq j \leq i$, and $\tau^j N_i$ is defined of finite dimension for $1 \leq i \leq n$ and $1 \leq j \leq 4$. Since M_2 is infinite dimensional by Lemma 3.2, $\text{rep}^+(Q)$ has an irreducible monomorphism $g : \tau M_3 \rightarrow M_2$, and by Statement (2), $d(\tau M_3) < d(\tau M_2)$. Moreover, it follows from Corollary 2.7 that $d(\tau^3 M_3) < d(\tau^2 M_2)$. This yields

$$d(\tau^3 M_3) + d(\tau M_3) < d(\tau^2 M_2) + d(\tau M_3) < d(\tau^2 M_2) + d(\tau M_2).$$

Suppose that $n \geq 2$. Since d is additive, $d(\tau^3 N_i) + d(\tau^2 N_i) \geq d(\tau^2 M_3)$ for $i = 1, 2$, and $d(\tau^2 M_3) + d(\tau M_3) \geq d(\tau^2 N_1) + d(\tau^2 N_2) + d(\tau M_2)$. Furthermore,

$$\begin{aligned} d(\tau^3 M_3) + d(\tau^2 M_3) &\geq d(\tau^3 N_1) + d(\tau^3 N_2) + d(\tau^2 M_2) \\ &\geq d(\tau^2 M_3) - d(\tau^2 N_1) + d(\tau^2 M_3) - d(\tau^2 N_2) + d(\tau^2 M_2) \\ &\geq d(\tau^2 M_3) + d(\tau M_2) - d(\tau M_3) + d(\tau^2 M_2). \end{aligned}$$

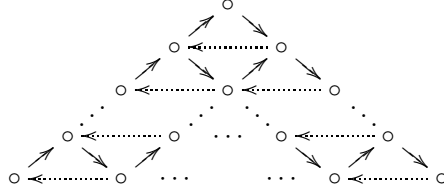
As a consequence, we get $d(\tau^3 M_3) + d(\tau M_3) \geq d(\tau^2 M_2) + d(\tau M_2)$, a contradiction. Thus N is indecomposable.

(5) Let M be not pseudo-projective. Then $\text{rep}^+(Q)$ has an almost split sequence

$$0 \longrightarrow \tau M \xrightarrow{(g_1, \dots, g_n)^T} N_1 \oplus \dots \oplus N_n \xrightarrow{(f_1, \dots, f_n)} M \longrightarrow 0,$$

where τM is finite dimensional and the N_i are indecomposable. Since $d(M) = \infty$, we may assume that $d(N_1) = \infty$. Then $g_1 : \tau M \rightarrow N_1$ is a monomorphism. By Statement (1), f_1 is an epimorphism. Hence, $g_i : \tau M \rightarrow N_i$ is an epimorphism, and hence $d(N_i) < \infty$, for $1 < i \leq n$. The proof of the lemma is completed.

Finally, we recall that a valued translation quiver is called a *wing* if it is isomorphic to the following trivially valued translation quiver:



where the dotted arrows indicate the translation; see [41, (3.3)].

Now we have the promised description of the regular components of $\Gamma_{\text{rep}^+(Q)}$.

4.14. THEOREM. *Let Q be an infinite connected strongly locally finite quiver, and let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$.*

- (1) *If Γ has no infinite dimensional or pseudo-projective representation, then it is of shape $\mathbb{Z}\mathbb{A}_\infty$.*
- (2) *If Γ has infinite dimensional but no pseudo-projective representations, then it is of shape $\mathbb{N}^- \mathbb{A}_\infty$ and its right-most section is a left infinite path.*
- (3) *If Γ has pseudo-projective but no infinite dimensional representations, then it is of shape $\mathbb{N}\mathbb{A}_\infty$ and its left-most section is a right infinite path.*
- (4) *If Γ has both pseudo-projective representations and infinite dimensional representations, then Γ is a finite wing.*

Proof. (1) Write $d(M) = \dim_k M \in \mathbb{N} \cup \{\infty\}$, for $M \in \Gamma$. Suppose that Γ contains no infinite dimensional or pseudo-projective representation. By Lemma 4.2, Γ is stable. Having no oriented cycle by Lemma 4.4, Γ is isomorphic to $\mathbb{Z}\Delta$, where Δ is a section of Γ ; see [32, (2.3)]. Consider the additive function

$$d : \Gamma_0 \rightarrow \mathbb{N} : M \mapsto d(M),$$

which is strictly monotone by Lemma 2.7, and consequently, Δ is either finite or of type \mathbb{A}_∞ ; see [42]. Suppose that Δ is finite. Let Θ be the full translation subquiver of Γ generated by the representations lying in $\Delta \cup \tau\Delta \cup \tau^{-1}\Delta$. Being connected and infinite, Θ has a finite connected full subquiver Σ , which contains the supports of the representations lying in Θ and which has more vertices than Δ does. Then Θ is a full translation subquiver of some connected component Γ' of $\Gamma_{\text{rep}(\Sigma)}$. Since Θ is finite and satisfies the condition *S4* stated in [32, (3.1)], Γ' is the preprojective or preinjective component of $\Gamma_{\text{rep}(\Sigma)}$ having Δ as a section. In particular, Σ and Δ have the same number of vertices, a contradiction. Thus, Δ is of type \mathbb{A}_∞ .

(2) Assume that Γ has infinite dimensional but no pseudo-projective representations. By Lemma 4.2(1), Γ is left stable, and by Lemma 4.5(1), the infinite dimensional representations lying in Γ generate a right-most section Δ . Therefore, $\Gamma \cong \mathbb{N}^-\Delta$; see [32, (2.3)]. By Lemma 4.13(2), Δ contains no right infinite path, and hence it has a sink-vertex M_0 . Having no pseudo-projective representation, by Lemma 4.13(5), Δ contains a left infinite path

$$(*) \quad \cdots \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0.$$

For each $n > 0$, denote by (d_n, d'_n) the valuation of the arrow $M_n \rightarrow M_{n-1}$. By Lemma 4.13(5), $d'_n = 1$ and M_n is the only immediate predecessor of M_{n-1} in Δ . By Lemma 4.13(4), $d_n = 1$ and M_{n-1} is the only immediate successor of M_n in Δ . This shows that the path $(*)$ is trivially valued and coincides with Δ . In particular, Γ is of shape $\mathbb{N}^-\mathbb{A}_\infty$.

(3) Assume that Γ has pseudo-projective but no infinite dimensional representations. By Proposition 4.12(2), Γ is the full translation subquiver of a right stable regular component Γ' of $\Gamma_{\text{rep}^-(Q)}$ obtained by deleting the infinite dimensional representations. By the dual of Statement (2), the infinite dimensional representations in Γ' generate a left-most section which is a right infinite path. As a consequence, the pseudo-projective representations in Γ generate a left-most section which is a right infinite path. In particular, Γ is of shape $\mathbb{N}\mathbb{A}_\infty$.

(4) Suppose that Γ has both infinite dimensional representations and pseudo-projective ones. By Lemma 4.5, Γ has a right-most section Δ generated by the infinite dimensional representations, and a left-most section generated by the pseudo-projective ones. In particular, Γ contains no left or right stable representation.

Next, we show that Δ is a finite trivially valued path. If Δ has no pseudo-projective representation, then it follows from Lemma 4.13(3),(5) that Δ has a left infinite path in which every representation is left stable, a contradiction. Thus Δ contains a pseudo-projective representation M_0 . By Lemma 4.13(2),(5), M_0 is a unique source vertex in Δ . If $N \in \Delta$ with $N \neq M_0$, then N is a proper successor of M_0 in Δ , and by Lemma 4.13(2), $\text{rep}^+(Q)$ has a chain of irreducible epimorphisms from M_0 to N . In particular, we have shown that $\text{supp}X \subseteq \text{supp}M_0$, for all $X \in \Delta$. Consider now an almost split sequence

$$\eta : \quad 0 \longrightarrow L \longrightarrow E \longrightarrow M_0 \longrightarrow 0$$

in $\text{rep}(Q)$, where $L \in \text{rep}^-(Q)$ is infinite dimensional. By Corollary 1.7, $\text{supp}L$ contains a left infinite path p with $e(p)$ lying in the socle-support of L . By Lemma 2.11, $e(p)$ is a successor in Q of some vertex in $\text{supp}M_0$. Not being contained in $\text{supp}M_0$ by Lemma 1.6(2), p passes through a vertex x which is not a successor of any vertex in $\text{supp}M_0$. Let Θ be the successor-closed subquiver of Q generated by x and $\text{supp}M_0$. By Proposition 2.17(1), $\text{rep}(\Theta)$ has an almost split sequence:

$$\eta_\Theta : 0 \longrightarrow L_\Theta \longrightarrow E_\Theta \longrightarrow M_0 \longrightarrow 0.$$

We define a new quiver Q' by attaching to Θ a right infinite path

$$u : x \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \xrightarrow{\alpha_3} \cdots,$$

where $a_i \notin Q$ for every $i \geq 1$. Let $X \in \Delta$. Since $\text{supp}X \subseteq \text{supp}M_0$, we have $X \in \text{rep}(\Theta)$ and $X \in \text{rep}(Q')$. Since x is not a successor in Q of any vertex in $\text{supp}M_0$, the vertices in $\text{supp}M_0$ have the same successors in Q and in Q' . This implies that $X \in \text{rep}^+(Q')$. Now, let $q : X \rightarrow Y$ with $X, Y \in \Delta$ be an irreducible morphism in $\text{rep}^+(Q)$. By Corollary 3.5, q is irreducible in $\text{rep}(Q)$, and hence irreducible in $\text{rep}(\Theta)$. Since Θ contains the successors in Q' of the vertices in $\text{supp}Y$, by Proposition 2.16(2), q is irreducible in $\text{rep}(Q')$, and hence irreducible in $\text{rep}^+(Q')$. This shows that Δ is a connected subquiver of $\Gamma_{\text{rep}^+(Q')}$. In particular, Δ is a subquiver of a connected component Γ' of $\Gamma_{\text{rep}^+(Q')}$.

Observe that x is a source vertex in Q' which is not a successor of any vertex in $\text{supp}M_0$. Thus Θ contains all the predecessors in Q' of the successors of the vertices in $\text{supp}M_0$. By Proposition 2.16(1), η_Θ is an almost split sequence in $\text{rep}(Q')$. Since L_Θ is finite dimensional, η_Θ is also an almost split sequence in $\text{rep}^+(Q')$. That is, $L_\Theta = \tau_\Theta M_0$. Furthermore, since Θ is top-finite, Q' has no left infinite path, and hence, $\text{rep}^-(Q') = \text{rep}^b(Q')$. In particular, $\Gamma_{\text{rep}^+(Q')}$ contains no pseudo-projective representation. Since $x \in \text{supp}L_\Theta$, applying Lemma 2.12 to the infinite acyclic walk u , we see that L_Θ is left stable in Γ' , and consequently, Γ' is not preprojective. Moreover, since M_0 is infinite dimensional, Γ' is not preinjective by Theorem 4.7. That is, Γ' is a regular component of $\Gamma_{\text{rep}^+(Q')}$. Since Γ' contains infinite dimensional but no pseudo-projective representations, by Statement (2), its infinite dimensional representations generate a right-most section Δ' , which is a trivially valued left infinite path. As a consequence, Γ' is trivially valued.

Let $X \rightarrow Y$ be an arrow in Δ whose valuation in Γ is (d, d') . Then $Y \neq M_0$, and by Lemma 4.13(2), Y is not pseudo-projective. In view of Lemma 4.13(5), we see that $d' = 1$. Suppose that $d > 1$. Then $\text{rep}^+(Q)$ has an irreducible morphism $f : X \rightarrow N$ with $N = Y \oplus Y$, which is also an irreducible morphism in $\text{rep}^+(\Theta)$. We claim that f is irreducible in $\text{rep}^+(Q')$. Indeed, let $f = hg$, where $g : X \rightarrow M$ and $h : M \rightarrow N$ are morphisms in $\text{rep}^+(Q')$. Since Θ is predecessor-closed in Q' , by Lemma 2.15(1), $M_\Theta \in \text{rep}^+(\Theta)$. This yields a factorization $f = h_\Theta g_\Theta$ in $\text{rep}^+(\Theta)$. Therefore, g_Θ is a section or h_Θ is a retraction. In the first case, $v'g_\Theta = \mathbf{1}_X$ for some morphism $v' : M_\Theta \rightarrow X$. Since Θ contains the predecessors in Q' of the vertices in $\text{supp}X$, we can extend v' to a morphism $v : M \rightarrow X$ in $\text{rep}(Q')$ such that $vg = \mathbf{1}_X$. That is, g is a section. Dually, if h_Θ is a retraction, then h is a retraction, since Θ contains the successors of the vertices in $\text{supp}N$. This establishes the claim. As a consequence, the arrow $X \rightarrow Y$ in Γ' has a non-trivial valuation, a contradiction. This proves that Δ is trivially valued as a valued subquiver of Γ . Furthermore, since the representations in Δ are all infinite dimensional, Δ is a full subquiver of

Δ' . Having a source vertex, Δ is of the form

$$M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{n-1} \longrightarrow M_n.$$

If $n = 0$, then $\Delta = \{M_0\}$, and consequently $\Gamma = \{M_0\}$, and we are done. Suppose now that $n > 0$. By Lemma 4.13(2), Γ contains a path

$$p: \quad \tau^n M_n \longrightarrow \tau^{n-1} M_{n-1} \longrightarrow \cdots \longrightarrow \tau M_1 \longrightarrow M_0.$$

Since M_0 is pseudo-projective, by Lemma 4.5(2), p contains only pseudo-projective representations. Moreover, p meets every τ -orbit in Γ since so does Δ . Thus, p is the left-most section of Γ generated by its pseudo-projective representations. This shows that Γ is a finite wing. The proof of the theorem is completed.

REMARK. (1) By Theorem 4.14(4), we have a one-one correspondence between the infinite dimensional pseudo-projective representations in $\Gamma_{\text{rep}^+(Q)}$ and the finite regular components of $\Gamma_{\text{rep}^+(Q)}$.

(2) Let Γ be a non-trivial regular component of $\Gamma_{\text{rep}^+(Q)}$. By Theorem 4.14, Γ contains a unique non-trivial τ -orbit \mathcal{O} in which every representation that is not pseudo-projective is the ending term of an almost split sequence with an indecomposable middle term. The representations in \mathcal{O} are called *quasi-simple*. Moreover, each representation M in Γ has a unique sectional path

$$M = M_n \longrightarrow \cdots \longrightarrow M_1,$$

with $n \geq 1$ and M_1 quasi-simple. One calls n the *quasi-length* of M . For convenience, the only representation in any trivial regular component is also called *quasi-simple*.

Applying Theorems 4.7, 4.6 and 4.14, we get immediately the following result.

4.15. COROLLARY. *If Q is an infinite connected strongly locally finite quiver, then $\Gamma_{\text{rep}^+(Q)}$ has a symmetric valuation.*

In the next two sections, we shall see that each of the four types of regular components does occur. To conclude this section, we give some conditions on Q such that $\Gamma_{\text{rep}^+(Q)}$ has at most one type of regular components. We start with the case where Q has no infinite path.

4.16. COROLLARY. *Let Q be an infinite connected strongly locally finite quiver. If Q has no infinite path, then $\Gamma_{\text{rep}^+(Q)}$ consists of a preprojective component of shape $\mathbb{N}Q^{\text{op}}$, a preinjective component of shape $\mathbb{N}^-Q^{\text{op}}$, and some possible regular components of shape $\mathbb{Z}\mathbb{A}_\infty$.*

Proof. Assume that Q has no infinite path. Then $\text{rep}^+(Q) = \text{rep}^b(Q) = \text{rep}^-(Q)$. In particular, $\Gamma_{\text{rep}^+(Q)}$ has no infinite dimensional or pseudo-projective representation. Now the result follows immediately from Theorems 4.6(1), 4.7(1), and 4.14(1). The proof of the corollary is completed.

Finally, for convenience, we shall call a non-trivial walk in Q an *almost-path* if all but finitely many of its edges are arrows.

4.17. THEOREM. *Let Q be an infinite connected strongly locally finite quiver.*

- (1) If every right infinite acyclic walk in Q is an almost-path, then every regular component of $\Gamma_{\text{rep}^+(Q)}$ is of shape $\mathbb{N}^-\mathbb{A}_\infty$.
- (2) If every left infinite acyclic walk in Q is an almost-path, then every regular component of $\Gamma_{\text{rep}^+(Q)}$ is of shape $\mathbb{N}\mathbb{A}_\infty$.

Proof. (1) Suppose that the right infinite acyclic walks in Q are all almost-paths. Since the inverse of a left infinite path is a right infinite acyclic walk which is not an almost-path, Q contains no left infinite path. Hence, $\text{rep}^-(Q) = \text{rep}^b(Q)$, and in particular, $\Gamma_{\text{rep}^+(Q)}$ has no pseudo-projective representation. Let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$. By Theorem 4.14, Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{N}^-\mathbb{A}_\infty$.

Suppose that Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$. Then, by Theorem 4.14 again, the representations in Γ are all finite dimensional. Fix arbitrarily a representation M in Γ . Observe that $\text{supp}(M \oplus \tau^{-1}M)$ is connected since $\text{Ext}^1(\tau^{-1}M, M) \neq 0$. As a consequence, $\Sigma = \text{supp}(\bigoplus_{i \geq 0} \tau^{-i}M)$ is connected. We claim that Σ is finite. Indeed, if Σ is infinite, then it contains a right infinite acyclic walk w , which is an almost-path by hypothesis. Write $w = vu$, where u is a finite walk and v is a right infinite path:

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

Observe that $a_0 \in \text{supp} \tau^{-r}M$ for some $r \geq 0$. Since $M \oplus \cdots \oplus \tau^{-r}M$ is finite dimensional, there exists a maximal integer s such that $a_s \in \text{supp}(M \oplus \cdots \oplus \tau^{-r}M)$. Then a_{s+1} lies in the support of some $\tau^{-j}M$ with $j > r$. Let t be minimal such that the support of $\tau^{-t}M$ contains some of the a_i with $i > s$. Then $t > r$. Since $\tau^{-t}M$ is finite dimensional, there exists a maximal integer l such that $a_l \in \text{supp} \tau^{-t}M$. Then $l > s$. Since $a_{l+1} \notin \text{supp} \tau^{-t}M$ and $a_l \rightarrow a_{l+1}$ is an arrow, by Corollary 2.11, $a_{l+1} \in \text{supp} \tau^{-(t-1)}M$, which is contrary to the minimality of t . Our claim is established. As a consequence, there exist two distinct integers $m, n \geq 0$ such that $\tau^{-m}M$ and $\tau^{-n}M$ have the same support, a contradiction to Proposition 2.14(2). Therefore, Γ is of shape $\mathbb{N}^-\mathbb{A}_\infty$.

(2) Suppose that the left infinite acyclic walks in Q are all almost-paths. Then Q has no right infinite path, and hence $\text{rep}^+(Q) = \text{rep}^b(Q)$. Hence $\Gamma_{\text{rep}^+(Q)}$ has no infinite dimensional representation. Let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$. By Theorem 4.14, Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{N}\mathbb{A}_\infty$. If Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$, then every representation in Γ is finite dimensional. Hence Γ is a regular component of $\Gamma_{\text{rep}^-(Q)}$. On the other hand, by the dual of Statement (1), all the regular components of $\Gamma_{\text{rep}^-(Q)}$ are of shape $\mathbb{N}\mathbb{A}_\infty$, a contradiction. Thus Γ is of shape $\mathbb{N}\mathbb{A}_\infty$. The proof of the theorem is completed.

REMARK. If Q is constructed from a finite quiver by attaching finitely many disjoint right infinite paths, then it clearly satisfies the condition stated in Theorem 4.17(1). Note, however, that the quiver

$$\begin{array}{ccccccc} \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \circ & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \circ & & \circ & & \circ & & \circ & & \end{array}$$

satisfies the stated condition, but it can not be constructed in this way.

5. REPRESENTATIONS OF INFINITE DYNKIN QUIVERS

Throughout this section, let Q stand for an infinite Dynkin quiver, that is, its underlying graph is one of the following three infinite diagrams:

$$\begin{array}{l}
 \mathbb{A}_\infty : \quad \cdots - \circ - \circ - \circ - \circ - \cdots \\
 \mathbb{A}_\infty : \quad \circ - \circ - \circ - \circ - \cdots \\
 \mathbb{D}_\infty : \quad \circ - \circ - \circ - \circ - \cdots \\
 \qquad \qquad \quad | \\
 \qquad \qquad \quad \circ
 \end{array}$$

As main results, we shall give a complete list of the non-isomorphic indecomposable representations in $\text{rep}^+(Q)$ and describe explicitly its Auslander-Reiten components. Note that Reiten and Van den Bergh have done so (with no proof) for each type of infinite Dynkin quivers with the alternating orientation; see [39, (III.3)]. More partial results in this subject can be found in [15, 36, 37].

As usual, some combinatorial consideration is needed. Let w be a reduced walk in Q . Denote by $Q(w)$ the full subquiver of Q generated by the vertices appearing in w . We say that w has no left infinite path or no right infinite path if so does $Q(w)$. Now, w is called a *string* if the quiver $Q(w)$ contains at least one, and at most finitely many, sink or source vertices. If w is a non-trivial string, then neither w nor w^{-1} is a double infinite path, and we can write uniquely w as $w = w_n \cdots w_1$, where w_1, \dots, w_n are non-trivial paths or inverses of non-trivial paths such that $w_{i+1}w_i$ is neither a path nor the inverse of a path, for $1 \leq i < n$. In this case, we call w_1 the *initial walk*, and w_n the *terminal walk*, of w . Let v, w be strings. In case $e(v) = s(w)$, we define the *composite* of v, w to be vw if it is a non-trivial reduced walk, or w if v is trivial, or v if w is trivial. For instance, if $\alpha : x \rightarrow y$ is an arrow, then $\alpha^{-1}\varepsilon_y = \alpha^{-1} = \varepsilon_x\alpha^{-1}$, but $\varepsilon_x \neq \alpha^{-1}\alpha$, since α, α^{-1} are not composable as strings. Now, v is called a *substring* of w if $w = svr$, where r, s are strings.

5.1. DEFINITION. Let Q be a quiver of type \mathbb{A}_∞ or \mathbb{A}_∞ . Each arrow $\alpha : y \leftarrow x$ determines a unique triple (q, α, p) , called a *double-hook*, where q is the longest path ending in y but not with α , and p is the longest path starting in x but not with α .

REMARK. A double-hook (q, α, p) has no left infinite path if and only if q is finite. In this case, $p\alpha^{-1}q$ is a string with no left infinite path.

EXAMPLE. Let Q be a quiver of type \mathbb{A}_∞ as follows:

$$0 \xrightarrow{\alpha} 1 \xrightarrow{\beta} 2 \xleftarrow{\gamma} 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow \cdots$$

Then $(\varepsilon_1, \alpha, \varepsilon_0)$ and $(\beta\alpha, \gamma, p_\infty)$ are double-hooks, where p_∞ denotes the infinite path starting in 3.

For convenience, we shall say that Q is *canonical* of type \mathbb{A}_∞ if $Q_0 = \mathbb{N}$ and the edges are of the form $x \rightarrow x + 1$; *canonical* of type \mathbb{A}_∞ if $Q_0 = \mathbb{Z}$ and the edges are of the form $x \rightarrow x + 1$; and *canonical* of type \mathbb{D}_∞ if $Q_0 = \mathbb{N}$ such that 0, 1 are of weight one and 2 is of weight three and the edges not attached to 0 are of the form $x \rightarrow x + 1$.

5.2. DEFINITION. Let Q be a canonical infinite Dynkin quiver, and let \mathcal{S} be a set of paths having pairwise distinct starting points.

(1) For $p, q \in \mathcal{S}$, we define $p \preceq q$ if and only if $s(p) \leq s(q)$.

- (2) For $p, q \in \mathcal{S}$, we define $p = \sigma_{\mathcal{S}}(q)$ and $q = \sigma_{\mathcal{S}}^-(p)$ if $p \prec q$ and there exists no u in \mathcal{S} such that $p \prec u \prec q$.
- (3) We call $\sigma_{\mathcal{S}}$ the *source-translation* in \mathcal{S} .

REMARK. (1) The relation \preceq is a well order in \mathcal{S} .

- (2) If $p, q \in \mathcal{S}$, then $p \prec q$ if and only if $p = \sigma^i(q)$ for some $i > 0$.

Let Q be canonical of type \mathbb{A}_{∞} or $\mathbb{A}_{\infty}^{\infty}$. A non-trivial path in Q is called *right-oriented* or *left-oriented* if its arrows are all of the form $x \rightarrow x+1$ or all of the form $x-1 \leftarrow x$, respectively. Moreover, a string w is called *normalized* if $s(u) \leq e(u)$ for any finite substring u of w . It is evident that w or w^{-1} is normalized.

5.3. NOTATION. Suppose that Q is a canonical quiver of type \mathbb{A}_{∞} or $\mathbb{A}_{\infty}^{\infty}$.

- (1) Let Q_R denote the set of right-oriented maximal paths having a starting point and the trivial paths ε_x , where x is either a middle point of a left-oriented path or a sink vertex of weight one.
- (2) Let Q_L denote the set of left-oriented maximal paths having a starting point and the trivial paths ε_x , where x is either a middle point of a right-oriented path or a source vertex of weight one.

We state some alternative defining properties of the paths in Q_R .

5.4. LEMMA. *Let Q be a canonical quiver of type \mathbb{A}_{∞} or $\mathbb{A}_{\infty}^{\infty}$.*

- (1) *If $x \in Q_0$ and p is a path, then $p \in Q_R$ with $s(p) = x+1$ if and only if $x \leftarrow x+1$ is an arrow, and p is the longest path starting in $x+1$ but not with $x \leftarrow x+1$.*
- (2) *If q is a finite path, then $q \in Q_R$ with $e(q) = x$ if and only if $x \leftarrow x+1$ is an arrow, and q is the longest path ending in x but not with $x \leftarrow x+1$.*

Proof. (1) Let $x \in Q_0$ and p be a path. Suppose that $\alpha : x \leftarrow x+1$ is an arrow, and p is the longest path starting in $x+1$ but not with α . If Q has an arrow $x+1 \leftarrow x+2$, then $p = \varepsilon_{x+1} \in Q_R$. If Q has an arrow $\gamma : x+1 \rightarrow x+2$, then p is the maximal path starting with γ . Being right-oriented, p lies in Q_R . Conversely, suppose that $p \in Q_R$ with $s(p) = x+1$. Assume that p is non-trivial. Then p is a right-oriented maximal path starting in $x+1$. Hence, Q has an arrow $\alpha : x \leftarrow x+1$. Being maximal and right-oriented, p is the longest path starting in $x+1$ but not with α . Otherwise, $p = \varepsilon_{x+1}$ with $x+1$ the middle point of a left-oriented path or a sink vertex of weight one. Since $x \in Q_0$, the second case does not occur. That is, Q has a path $x \leftarrow x+1 \leftarrow x+2$. In this situation, ε_{x+1} is the longest path starting in $x+1$ but not with α .

(2) Let q be a finite path in Q . Suppose that $\alpha : x \leftarrow x+1$ is an arrow, and q is the longest path ending in x but not with α . If x is a sink vertex of weight one or $x-1 \leftarrow x$ is an arrow, then $q = \varepsilon_x \in Q_R$. Otherwise, $\delta : x-1 \rightarrow x$ is an arrow, and q is the maximal path ending with δ . Being right-oriented and a maximal path, $q \in Q_R$. Conversely, suppose that $q \in Q_R$ with $e(q) = x$. Assume that q is non-trivial. Then it is a right-oriented maximal path ending in x . Hence, $\alpha : x \leftarrow x+1$ is an arrow. Being maximal and right-oriented, q is the longest path ending in x but not with α . Otherwise, $q = \varepsilon_x$ with x the middle point of a left-oriented path or a sink vertex of weight one. In either case, $\alpha : x \leftarrow x+1$ is an arrow and ε_x is the longest path ending in x but not with α . The proof of the lemma is completed.

Similarly, we have some alternative defining properties of the paths in Q_L .

5.5. LEMMA. *Let Q be a canonical quiver of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ .*

- (1) *If $x \in Q_0$ and p is a path, then $p \in Q_L$ with $s(p) = x$ if and only if $x \rightarrow x + 1$ is an arrow, and p is the longest path starting in x but not with $x \rightarrow x + 1$.*
- (2) *If $x \in Q_0$ and q is a finite path, then $q \in Q_L$ with $e(q) = x + 1$ if and only if $x \rightarrow x + 1$ is an arrow, and q is the longest path ending in $x + 1$ but not with $x \rightarrow x + 1$.*

Let Q be canonical of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ . Since Q_R and Q_L are sets of paths having pairwise distinct starting points, they are equipped with the well order \preceq and the source-translation which is denoted by σ_R for Q_R and by σ_L for Q_L . If no risk of confusion is possible, the subscripts in σ_R and σ_L will be dropped. The following result reveals the link between the source-translates and the double-hooks.

5.6. LEMMA. *Let Q be a canonical quiver of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ .*

- (1) *If p, q are paths in Q , then $p, q \in Q_R$ with $q = \sigma_R(p)$ if and only if q is a finite path in Q_R and $\alpha : e(q) \leftarrow s(p)$ is an arrow such that (q, α, p) is a double-hook.*
- (2) *If p, q are paths in Q , then $p, q \in Q_L$ with $q = \sigma_L^-(p)$ if and only if q is a finite path in Q_L and $\beta : e(q) \leftarrow s(p)$ is an arrow such that (q, β, p) is a double-hook.*
- (3) *If (q, α, p) is a double-hook with q finite, then $p, q \in Q_R$ or $p, q \in Q_L$.*

Proof. Let p, q be paths in Q . Suppose that q is a finite path in Q_R with $e(q) = x$ and $\alpha : x \leftarrow s(p)$ is an arrow such that (q, α, p) is a double-hook. By Lemma 5.4(2), $s(p) = x + 1$, and by Lemma 5.4(1), $p \in Q_R$. Since $s(q) \leq x < x + 1 = s(p)$, we have $q \prec p$. If q is trivial, then $q = \varepsilon_x$ with $s(q) = s(p) - 1$, and hence $q = \sigma_R(p)$. Otherwise, q is a right-oriented maximal path ending in x . Thus, for any $y \in Q_0$ with $s(q) < y \leq x$, we have an arrow $y - 1 \rightarrow y$, and by Lemma 5.4(1), $y \neq s(v)$, for any $v \in Q_R$. Thus, $\sigma_R(p) = q$.

Conversely, suppose that $p, q \in Q_R$ with $q = \sigma_R(p)$. Write $b = s(q) < s(p) = a$. In particular, $a - 1 \in Q_0$. By Lemma 5.4(1), $\alpha : a - 1 \leftarrow a$ is an arrow, and p is the longest path starting in a but not with α . Let (u, α, p) be the double-hook determined by α , that is, u is the longest path ending in $a - 1$ but not with α . If u is infinite, then it is right-oriented and contains the arrow $b - 1 \rightarrow b$, which contradicts Lemma 5.4(1) since $q \in Q_R$. Thus u is finite. By the sufficiency we have proved, $u = \sigma_R(p) = q$. This establishes Statement (1). Similarly, we may prove Statement (2).

Finally, let (q, α, p) be a double-hook with q finite. If α is a left-oriented arrow $x \leftarrow x + 1$, then $p, q \in Q_R$ by Lemma 5.4. If α is a right-oriented arrow $x \rightarrow x + 1$, then $p, q \in Q_L$ by Lemma 5.5. The proof of the lemma is completed.

5.7. COROLLARY. *Let Q be a canonical quiver of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ .*

- (1) *If $q \in Q_R$, then $\sigma_R^-(q)$ is defined in Q_R if and only if q is finite.*
- (2) *If $p \in Q_L$, then $\sigma_L(p)$ is defined in Q_L if and only if p is finite with $e(p) - 1 \in Q_0$.*

Proof. Let $q \in Q_R$. If $\sigma_R^-(q) = p \in Q_R$, then $q = \sigma_R(p)$, which is finite by Lemma 5.6(1). Conversely, suppose that q is finite with $e(q) = x$. By Lemma 5.4(2), $\alpha : x \leftarrow x + 1$ is an arrow, and q is the longest path ending in x but not with α . Let (q, α, p) be the double-hook determined by α . By Lemma 5.6(1), $p \in Q_R$ and

$q = \sigma_R(p)$. That is, $p = \sigma_R^-(q)$. This establishes Statement (1). Similarly, we can prove Statement (2). The proof of the corollary is completed.

We are now ready to study the representation theory of Q . First, we show that, as in the finite Dynkin case, the indecomposable representations in $\text{rep}^+(Q)$ are uniquely determined by their dimension vector.

5.8. PROPOSITION. *Let Q be an infinite Dynkin quiver. If $M, N \in \text{rep}^+(Q)$ are indecomposable, then $M \cong N$ if and only if $\dim M(x) = \dim N(x)$, for all $x \in Q_0$.*

Proof. Let $M, N \in \text{rep}^+(Q)$ be indecomposable such that $\dim M(x) = \dim N(x)$, for all $x \in Q_0$. In particular, M, N have the same support which is denoted as Q' . By Theorem 1.12(1), $M \oplus N$ is projective restricted to a co-finite successor-closed subquiver Σ of Q' . Let Ω be the predecessor-closed subquiver of Q' generated by the augmented complement of Σ in Q' and the top-support of $(M \oplus N)_x$. By the dual of Theorem 1.13(2), M_Ω and N_Ω are indecomposable. Since Q' is top-finite and Σ is co-finite in Q' , we see that Ω is finite. Since Q is of infinite Dynkin type, Ω is a finite Dynkin quiver. Therefore, $M_\Omega \cong N_\Omega$, and hence $M \cong N$ by the dual of Theorem 1.13(1). The proof of the proposition is completed.

In order to classify the indecomposable representations in $\text{rep}^+(Q)$ we need to define, for each string w in Q , the *string* representation $M(w)$ as follows: if $x \in Q_0$, then $M(w)(x) = k$ in case x appears in w and $M(w)(x) = 0$ otherwise; and if $\alpha \in Q_1$, then $M(w)(\alpha) = \mathbf{1}$ in case α or α^{-1} appears in w , and $M(w)(\alpha) = 0$ otherwise; compare [13, page 158]. By definition, $M(w) = M(w^{-1})$. In case Q is of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ , we shall prove that the indecomposable representations in $\text{rep}^+(Q)$ are parameterized by the strings having no left infinite path. Note that this does not follow directly from the result of Butler and Ringel stated in [13]. Indeed, if Q contains infinite paths, then the path algebra kQ is not a string algebra as defined in [13, Section 3]. Nevertheless, Theorem 1.13 allows us to apply their results.

5.9. PROPOSITION. *Let Q be a quiver of type \mathbb{A}_∞^∞ or \mathbb{A}_∞ . If $M \in \text{rep}(Q)$ is indecomposable, then M is finitely presented if and only if $M \cong M(w)$ with w a string having no left infinite path.*

Proof. We may assume that Q is canonical. For proving the sufficiency, let w be a string in Q having no left infinite path, which we may assume to be infinite and normalized. Let w_1 be the initial walk and w_n the terminal walk of w . Suppose that w_1, w_n are distinct and both infinite. Then w_1^{-1} and w_n are right infinite paths. Observe that $\text{supp} M(w) = Q$. Let $\alpha : x \rightarrow y$ be the initial arrow of w_1^{-1} and $\beta : a \rightarrow b$ the initial arrow of w_n . Consider the full subquiver Σ of Q generated by the successors of y and those of b . Then Σ is successor-closed and co-finite in Q such that $M(w)_x = P_y \oplus P_b$. By Theorem 1.12(1), $M(w) \in \text{rep}^+(Q)$. In case either w_1 or w_n is infinite, we can prove in a similar manner that $M(w) \in \text{rep}^+(Q)$.

Conversely, suppose that M is an indecomposable representation in $\text{rep}^+(Q)$. Then $\text{supp}(M)$ is connected, and hence, $\text{supp}(M) = Q(\omega)$, for some reduced walk ω . Being top-finite, $\text{supp}(M)$ has at most finitely many source vertices and no left infinite path. That is, ω is a string having no left infinite path. By Theorem 1.12(1), M is projective restricted to a co-finite successor-closed subquiver Σ of $\text{supp}(M)$. Fix a vertex x in $\text{supp}(M)$. Let Ω be the predecessor-closed subquiver of $\text{supp}(M)$ generated by x , the augmented complement of Σ in $\text{supp}(M)$ and the top-support

of M_x , which is finite. By Theorem 1.13(2), M_Ω is indecomposable. Then, by the theorem stated in [13, Section 3], M_Ω is a string representation of Ω . In particular, $\dim_k M(x) = 1$. Since $\text{supp}(M) = Q(\omega) = \text{supp}M(\omega)$, by Lemma 5.8, $M \cong M(\omega)$. The proof of the proposition is completed.

We describe the irreducible morphisms in $\text{rep}^+(Q)$ in the following proposition; compare [13, page 166].

5.10. PROPOSITION. *Let Q be a quiver of type \mathbb{A}_∞^∞ or \mathbb{A}_∞ , and let α be an arrow and w a string having no left infinite path in Q .*

- (1) *If p is a path of maximal length such that $w\alpha p^{-1}$ or $p\alpha^{-1}w$ is a string, then the canonical embedding $M(w) \rightarrow M(w\alpha p^{-1})$ or $M(w) \rightarrow M(p\alpha^{-1}w)$ is irreducible in $\text{rep}^+(Q)$, respectively.*
- (2) *If q is a finite path of maximal length such that $w\alpha^{-1}q$ or $q^{-1}\alpha w$ is a string, then the canonical projection $M(w\alpha^{-1}q) \rightarrow M(w)$ or $M(q^{-1}\alpha w) \rightarrow M(w)$ is irreducible in $\text{rep}^+(Q)$, respectively.*

Proof. We shall prove the proposition only for one case, since the other cases can be treated similarly. Suppose that Q is canonical and w is normalized such that $w\alpha$ is a string. Let p be the longest path such that $v = w\alpha p^{-1}$ is a string. Since w has no left infinite path, nor does v . The canonical embedding $f : M(w) \rightarrow M(v)$ is defined so that $f(x) = \mathbf{1}$ if x is a vertex lying in w ; and otherwise, $f(x) = 0$.

Suppose that $f = hg$, where $g : M(w) \rightarrow N$ and $h : N \rightarrow M(v)$ are morphisms in $\text{rep}^+(Q)$. It suffices to prove that g is a section or h is a retraction. For this purpose, we may assume that $\text{Hom}(M, M(v)) \neq 0$ for any indecomposable summand M of N . Write $L = M(w) \oplus N \oplus M(v)$, and consider $Q' = \text{supp} L$, which is connected by the assumption. By Theorem 1.12(1), L is projective restricted to a co-finite successor-closed subquiver Σ of Q' . Being connected, Q' has a finite connected subquiver Δ which contains the arrow α , the top-support of L_x and the augmented complement of Σ in Q' . Since Q' is top-finite, the predecessor-closed subquiver Ω of Q' generated by Δ is finite and connected. Then $M(w)_\Omega = M(u)$ and $M(v)_\Omega = M(u\alpha q^{-1})$, where $u = w \cap \Omega$ and $q = p \cap \Omega$. Observe that q is the longest path in Ω such that $u\alpha q^{-1}$ is a string in Ω , and $f_\Omega : M(u) \rightarrow M(u\alpha q^{-1})$ is the canonical embedding. By the lemma stated in [13, page 166], f_Ω is irreducible in $\text{rep}(\Omega)$. Thus g_Ω is a section or h_Ω is a retraction. By the dual of Theorem 1.13(3), g is a section or h is a retraction. The proof of the proposition is completed.

5.11. LEMMA. *Let Q be a quiver of type \mathbb{A}_∞^∞ or \mathbb{A}_∞ , and let w be a string such that $M(w)$ and $\text{DTr}M(w)$ are indecomposable objects in $\text{rep}^+(Q)$. If α is an arrow and q is a path such that $w\alpha^{-1}q$ is a string, then q is finite.*

Proof. Let α be an arrow and q be a path such that $w\alpha^{-1}q$ is a string. Then $M(w)$ admits a minimal projective resolution

$$0 \longrightarrow \bigoplus_{i=1}^r P_{x_i} \longrightarrow \bigoplus_{j=1}^s P_{y_j} \longrightarrow M(w) \longrightarrow 0,$$

where y_1, \dots, y_s are the source vertices in $Q(w)$, and by Lemma 2.10, we may assume that $x_1 = e(q)$. By Lemma 2.5, $\text{DTr}M(w)$ admits a minimal injective co-resolution

$$0 \longrightarrow \text{DTr}M(w) \longrightarrow \bigoplus_{i=1}^r I_{x_i} \longrightarrow \bigoplus_{j=1}^s I_{y_j} \longrightarrow 0.$$

Since $x_1 \notin Q(w)$, none of the vertices in $\text{supp}(\oplus_{j=1}^s I_{y_j})$ appears in q . Hence, q is contained in the support of $\text{DTr}M(w)$. By Lemma 1.6(2), q is not a left infinite path, and hence it is finite. The proof of the lemma is completed.

As shown below, the almost split sequences with an indecomposable middle term in $\text{rep}^+(Q)$ are parameterized by the double-hooks with no left infinite path; compare [13, page 174].

5.12. PROPOSITION. *Let Q be an infinite Dynkin quiver of type \mathbb{A}_∞ or \mathbb{A}_∞ .*

(1) *If (q, α, p) is double-hook with q finite, then $\text{rep}^+(Q)$ has an almost split sequence*

$$0 \longrightarrow M(q) \longrightarrow M(p\alpha^{-1}q) \longrightarrow M(p) \longrightarrow 0.$$

(2) *Every almost split sequence with an indecomposable middle term in $\text{rep}^+(Q)$ is of the form stated above.*

Proof. Let (q, α, p) be double-hook with q finite. Consider the short exact sequence

$$(1) \quad 0 \longrightarrow M(q) \xrightarrow{f} M(p\alpha^{-1}q) \xrightarrow{g} M(p) \longrightarrow 0$$

in $\text{rep}^+(Q)$, where f is the canonical embedding and g is the canonical projection. By Proposition 5.10, f and g are irreducible. Since $\text{rep}^+(Q)$ is a Krull-Schmidt category, the sequence (1) is almost split; see [6, (2.15)]. Conversely, assume that

$$(2) \quad 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

is an almost split sequence in $\text{rep}^+(Q)$ with M indecomposable. By Proposition 5.9, $N = M(w)$, where w is a string with no left infinite path. We shall show that the sequence (2) is as stated in Statement (1) by considering all possible cases.

First, assume that there exists an arrow β such that $w\beta^{-1}$ is a string. Let q be the longest path such that $w\beta^{-1}q$ is a string. By Lemma 5.11, q is finite. Now $\text{rep}^+(Q)$ has a short exact sequence

$$(3) \quad 0 \longrightarrow M(q) \xrightarrow{f'} M(w\beta^{-1}q) \xrightarrow{g'} M(w) \longrightarrow 0,$$

where f' is the canonical embedding and g' is the canonical projection. By Lemma 5.9(2), g' is irreducible. Since M is indecomposable, the sequence (3) is isomorphic to the sequence (2), and hence it is almost split. Let p be the longest path such that $p\beta^{-1}q$ is a string. Then (q, α, p) is a double-hook with q finite. By the sufficiency we have just proved,

$$0 \longrightarrow M(q) \longrightarrow M(p\beta^{-1}q) \longrightarrow M(p) \longrightarrow 0$$

is an almost split sequence in $\text{rep}^+(Q)$, which is isomorphic to the sequence (3). That is, the sequence (2) is of the desired form. Similarly, we can treat the case where there exists an arrow β such that βw is a string.

Next, suppose that there exists an arrow γ such that $\gamma^{-1}w$ is a string. Let p be the longest path such that $p\gamma^{-1}w$ is a string. Consider the short exact sequence

$$(4) \quad 0 \longrightarrow M(w) \xrightarrow{u} M(p\gamma^{-1}w) \xrightarrow{v} M(p) \longrightarrow 0$$

in $\text{rep}^+(Q)$, where u is the canonical embedding and v is the canonical projection. By Lemma 5.10(1), u is irreducible. In particular, $M(w)$ is not injective. Since $M(w)$ is not projective, neither is $M(p\gamma^{-1}w)$. Since M is indecomposable, $\tau M(p\gamma^{-1}w) = M$. By Proposition 3.6, M is of finite dimension, and so is $M(w)$. Therefore, $\text{rep}^+(Q)$ has an almost split sequence

$$(5) \quad 0 \longrightarrow M(w) \longrightarrow E \longrightarrow \text{Tr}DM(w) \longrightarrow 0.$$

Since $M(w)$ is not projective, E has no projective direct summand, and since M is indecomposable, so is E . As a consequence, the sequence (4) is isomorphic to the sequence (5), and hence it is almost split. Let q be the longest path such that $p\gamma^{-1}q$ is a string. By Lemma 5.11, q is finite, and consequently, $\text{rep}^+(Q)$ has an almost split sequence

$$0 \longrightarrow M(q) \longrightarrow M(p\beta^{-1}q) \longrightarrow M(p) \longrightarrow 0,$$

which is isomorphic to the sequence (4). In particular, $w = q$. Since $M(q) = M(w)$ is not projective, there exists some arrow β such that $q\beta^{-1}$ is a string. This turns out to be the first case we have treated. Thus, the sequence (2) is of the desired form. Similarly, we can deal with the case where there exists an arrow γ such that $w\gamma$ is a string.

Now, we consider the case where $e(w)$ is not defined. Since w is a string with no left infinite path, we may write $w = pv$, where v is a string and p is a right infinite path with $s(p)$ a source vertex in $Q(w)$. If v is not trivial, then we can write $w = p\beta^{-1}u$, where u is a string and β is an arrow. By Lemma 5.10(1), there exists an irreducible monomorphism $j : M(u) \rightarrow M(w)$, which is impossible since M is indecomposable. Thus $w = p$. Since $M(p)$ is not projective, there exists an arrow β such that $p\beta^{-1}$ is a string. This is again the first case we have treated. Finally, we can similarly treat the case where $s(w)$ is not defined. The proof of the proposition is completed.

We shall now describe the Auslander-Reiten components of $\text{rep}^+(Q)$ in case Q is of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ . By Lemma 5.9, we may choose the vertex set of $\Gamma_{\text{rep}^+(Q)}$ to be the set of the finitely presented string representations.

5.13. LEMMA. *Let Q be a canonical quiver of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ . If $p \in Q_R$, then*

- (1) $\tau M(p)$ is defined in $\Gamma_{\text{rep}^+(Q)}$ if and only if $\sigma(p)$ is defined in Q_R , and in this case, $\tau M(p) = M(\sigma(p))$;
- (2) $\tau^-M(p)$ is defined in $\Gamma_{\text{rep}^+(Q)}$ if and only if $\sigma^-(p)$ is defined in Q_R , and in this case, $\tau^-M(p) = M(\sigma^-(p))$.

Proof. (1) Let $p \in Q_R$ with $s(p) = x$. Suppose that $\tau M(p)$ is defined in $\Gamma_{\text{rep}^+(Q)}$. Since $M(p)$ is not projective, Q has an arrow $\alpha : x-1 \leftarrow x$. By Lemma 5.4(1), p is the longest path starting in x but not with α . Let q be the longest path ending in $x-1$ but not with α . By Lemmas 5.11 and 5.4(2), q is a finite path in Q_R . By Lemma 5.6(1), $q = \sigma(p)$. Conversely, suppose that $\sigma(p) = q \in Q_R$. By Lemma 5.6(1), q is finite and $\alpha : e(q) \leftarrow s(p)$ is an arrow such that (q, α, p) is a double-hook. By Proposition 5.12(1), $\tau M(p) = M(q)$.

(2) If $\sigma^-(p) = q \in Q_R$, then $p = \sigma(q)$, and hence $\tau^-M(p) = M(q)$ by Statement (1). Conversely, suppose that $\tau^-M(p) \in \Gamma_{\text{rep}^+(Q)}$. By Lemma 4.2(2), $M(p)$ is finite dimensional, that is, p is finite. By Corollary 5.7(1), $\sigma^-(p)$ is defined in Q_R . The proof of the lemma is completed.

Similarly, we have the following statement.

5.14. LEMMA. *Let Q be a canonical quiver of type \mathbb{A}_∞ or \mathbb{A}_∞^∞ . If $q \in Q_L$, then*

- (1) $\tau M(q)$ is defined $\Gamma_{\text{rep}^+(Q)}$ if and only if $\sigma^-(q)$ is defined in Q_L , and in this case, $\tau M(q) = M(\sigma^-(q))$;
- (2) $\tau^- M(q)$ is defined $\Gamma_{\text{rep}^+(Q)}$ if and only if $\sigma(q)$ is defined in Q_L , and in this case, $\tau^- M(q) = M(\sigma(q))$.

Let \mathcal{O} be a τ -orbit in $\Gamma_{\text{rep}^+(Q)}$. We shall say that \mathcal{O} is *preprojective*, *preinjective* or *regular* if it contains preprojective, preinjective or regular representations, respectively. Furthermore, \mathcal{O} is called *quasi-simple* if it consists of quasi-simple regular representations.

5.15. PROPOSITION. *Let Q be a canonical quiver of type \mathbb{A}_∞^∞ or \mathbb{A}_∞ , and write $\mathcal{O}_R = \{M(p) \mid p \in Q_R\}$ and $\mathcal{O}_L = \{M(p) \mid p \in Q_L\}$.*

- (1) *If Q_R is non-empty, then \mathcal{O}_R is a preprojective or quasi-simple τ -orbit in $\Gamma_{\text{rep}^+(Q)}$, where the second case occurs if and only if Q is of type \mathbb{A}_∞^∞ .*
- (2) *If Q_L is non-empty, then \mathcal{O}_L is a preinjective or regular τ -orbit in $\Gamma_{\text{rep}^+(Q)}$, where the second case occurs if and only if Q is of type \mathbb{A}_∞^∞ .*
- (3) *If Γ is a regular component of $\Gamma_{\text{rep}^+(Q)}$, then it contains either \mathcal{O}_R or \mathcal{O}_L but not both, and consequently, Q is of type \mathbb{A}_∞^∞ .*

Proof. It is evident that for any $p, q \in Q_R$, we have $q = \sigma^i(p)$ for some $i \in \mathbb{Z}$. Making use of Lemma 5.13, we deduce that \mathcal{O}_R is a τ -orbit in $\Gamma_{\text{rep}^+(Q)}$. Suppose that \mathcal{O}_R is preinjective. Then $M(p) = I_x$ for some $p \in Q_R$ and $x \in Q^+$. In particular, p is finite. By Corollary 5.7(1) and Lemma 5.13(2), $\tau^- M(p)$ is defined in $\Gamma_{\text{rep}^+(Q)}$, a contradiction. Thus, \mathcal{O}_R is preprojective or regular. Assume that the second case occurs. Suppose that \mathcal{O}_R is non-trivial. Let $M(p)$ with $p \in Q_R$ be not pseudo-projective. By Lemma 5.13(1), $q = \sigma(p) \in Q_R$. By Lemma 5.6(1), we have a double hook (q, α, p) with q finite. By Proposition 5.12, $M(p)$ is quasi-simple. That is, \mathcal{O}_R is quasi-simple. Suppose that now \mathcal{O}_R is regular, $\mathcal{O}_R = \{M(p)\}$, where $p \in Q_R$ and $M(p)$ is infinite-dimensional and pseudo-projective. In particular, p is infinite. Since $M(p)$ is not projective, Q contains an arrow $\alpha : s(p) - 1 \leftarrow s(p)$. Let (q, α, p) be a double hook in Q . By Lemma 5.6(1), q is infinite. As a consequence, $Q = Q(w)$ with $w = p\alpha^{-1}q$. In this case, there exists a canonical projection $f : M(w) \rightarrow M(p)$. Let $g : M(u) \rightarrow M(p)$ be a non-zero non-isomorphism in $\text{rep}^+(Q)$, where u is a string with no left infinite path. If $M(u)$ is projective, then u is a proper subpath of p , and in this case, g factors through f . Otherwise, $u = p\alpha^{-1}v$, where v is a finite subpath of q ending in $s(p) - 1$. Thus g factors through f . That is, f is not irreducible in $\text{rep}(Q)$, and by Corollary 3.5, it is not irreducible in $\text{rep}^+(Q)$. Thus, $\Gamma_{\text{rep}^+(Q)}$ has no arrow ending in $M(p)$. By Theorem 4.14, $\{M(p)\}$ is a trivial component of $\Gamma_{\text{rep}^+(Q)}$. In particular, \mathcal{O}_R is quasi-simple.

Let Q be of type \mathbb{A}_∞ . Then 0 is a sink or source vertex of weight one. In the first case, $\varepsilon_0 \in Q_R$ with $M(\varepsilon_0) = P_0$, and in the second case, the maximal path p_0 starting in 0 lies in Q_R and $M(p_0) = P_0$. In either case, \mathcal{O}_R is preprojective. Conversely, suppose that \mathcal{O}_R is preprojective. Then, $M(p) = P_x$ for some $p \in Q_R$ with $x = s(p)$. If $x - 1 \in Q_0$, then Q has an arrow $x - 1 \leftarrow x$, which is absurd since $M(p)$ is projective. Thus $x - 1 \notin Q_0$, that is, Q is of type \mathbb{A}_∞ . This establishes Statement (1). In a similar way, we can prove Statement (2).

To prove Statement (3), let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$. Suppose first that $\Gamma = \{M(w)\}$, where w is a normalized string with no left infinite path. By Theorem 4.14(4), $M(w)$ is infinite dimensional, and hence w is infinite. Then either

the initial walk w_1 of w or the terminal walk w_n is infinite. In the second case, being normalized and not the inverse of a left infinite path, w_n is a right-oriented maximal path with a starting point. In particular, $w_n \in Q_R$. If $w \neq w_n$, then $w = w_n \alpha^{-1} v$, where α is an arrow and v is a string. Since w_n is a maximal path, by Proposition 5.10(1), $\text{rep}^+(Q)$ has an irreducible morphism $M(v) \rightarrow M(w)$, a contradiction. Therefore, $w = w_n \in Q_R$, and hence $\Gamma = \mathcal{O}_R$. Similarly, if w_1 is infinite, then $\Gamma = \mathcal{O}_L$. Suppose now that Γ is non-trivial. By Theorem 4.14, $\text{rep}^+(Q)$ has an almost split sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$, where $L, M, N \in \Gamma$. By Proposition 5.12, there exists a double-hook (q, α, p) such that $L = M(q)$ and $N = M(p)$. By Lemma 5.6(3), $L, N \in \mathcal{O}_R$ or $L, N \in \mathcal{O}_L$. That is, Γ contains \mathcal{O}_R or \mathcal{O}_L . Finally, observe that $Q_R \cap Q_L = \emptyset$. Having only one quasi-simple τ -orbit, Γ does not contain \mathcal{O}_R and \mathcal{O}_L . The proof of the proposition is completed.

EXAMPLE. Let Q be a canonical quiver of type \mathbb{A}_∞ as follows :

$$\dots \longleftarrow 0 \longleftarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6 \longleftarrow 7 \longleftarrow \dots$$

We denote by $p_{i,j}$ the path from i to j , and by p_∞ the infinite path starting in 2. Then $Q_L = \{p_\infty, p_{4,3}, \varepsilon_5\}$ in which the action of σ is indicated as follows:

$$\varepsilon_5 \xrightarrow{\sigma} p_{4,3} \xrightarrow{\sigma} p_\infty.$$

By Lemma 5.14(1), the action of τ in \mathcal{O}_L is indicated as follows:

$$S_5 \xleftarrow{\tau} M(p_{4,3}) \xleftarrow{\tau} M(p_\infty).$$

On the other hand, $Q_R = \{\varepsilon_i \mid i \leq 1\} \cup \{p_{2,3}, p_{4,6}\} \cup \{\varepsilon_i \mid i \geq 7\}$, in which the action of σ is indicated as follows:

$$\dots \xleftarrow{\sigma} \varepsilon_{-1} \xleftarrow{\sigma} \varepsilon_0 \xleftarrow{\sigma} \varepsilon_1 \xleftarrow{\sigma} p_{2,3} \xleftarrow{\sigma} p_{4,6} \xleftarrow{\sigma} \varepsilon_7 \xleftarrow{\sigma} \varepsilon_8 \xleftarrow{\sigma} \dots$$

By Lemma 5.13(1), the action of τ in \mathcal{O}_R is indicated as follows:

$$\dots \xleftarrow{\tau} S_{-1} \xleftarrow{\tau} S_0 \xleftarrow{\tau} S_1 \xleftarrow{\tau} M(p_{2,3}) \xleftarrow{\tau} M(p_{4,6}) \xleftarrow{\tau} S_7 \xleftarrow{\tau} S_8 \xleftarrow{\tau} \dots$$

The following result describes the Auslander-Reiten components in the \mathbb{A}_∞ -case.

5.16. THEOREM. *Let Q be an infinite Dynkin quiver of type \mathbb{A}_∞ .*

- (1) *If Q has no left infinite path, then $\Gamma_{\text{rep}^+(Q)}$ consists of the preprojective component and a preinjective component of shape $\mathbb{N}^- \mathbb{A}_\infty$.*
- (2) *If Q is a left infinite path, then $\Gamma_{\text{rep}^+(Q)}$ consists of the preprojective component of shape $\mathbb{N} \mathbb{A}_\infty$.*
- (3) *If Q is not a left infinite path but has left infinite paths, then $\Gamma_{\text{rep}^+(Q)}$ consists of the preprojective component of shape $\mathbb{N} \mathbb{A}_\infty$ and a finite preinjective component.*

Proof. We may assume that Q is canonical. By Theorem 4.6 and Proposition 5.15, $\Gamma_{\text{rep}^+(Q)}$ has a unique preprojective component but no regular component. Moreover, by Theorem 4.7, the preinjective components correspond bijectively to the connected component of Q^+ . If Q has no left infinite path, then $Q^+ = Q$, which is connected. Thus $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component which is of shape $\mathbb{N}^- \mathbb{A}_\infty$ by Theorem 4.7(1).

Suppose now that Q contains left infinite paths. Then it has no right infinite path. By Theorem 4.6(1), the preprojective component is of shape $\mathbb{N} \mathbb{A}_\infty$. If Q is a left infinite path, then $Q^+ = \emptyset$, and hence $\Gamma_{\text{rep}^+(Q)}$ has no preinjective component.

Otherwise, Q has a left infinite maximal path with an ending point $x > 0$. Then x is a sink vertex, and Q^+ is generated by the vertices y with $0 \leq y < x$. In particular, Q^+ is finite and connected. By Theorem 4.7, $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component \mathcal{I} , which has a right-most section of shape $(Q^+)^{\text{op}}$ and contains only finite τ -orbits. In particular, \mathcal{I} is finite. The proof of the theorem is completed.

Next, we shall describe the Auslander-Reiten components of $\text{rep}^+(Q)$ in the \mathbb{A}_∞^∞ -case. Since the preprojective component and the possible preinjective components have been described in Theorems 4.6 and 4.7, we shall concentrate on the regular components.

5.17. THEOREM. *Let Q be a quiver of type \mathbb{A}_∞ , having r right infinite maximal paths and l left infinite maximal paths with $0 \leq l, r \leq 2$. Then $\Gamma_{\text{rep}^+(Q)}$ consists of the preprojective component, at most one preinjective component, and at most two regular components which are described as follows.*

- (1) *If Q is a double infinite path, then $\Gamma_{\text{rep}^+(Q)}$ has a unique regular component of shape $\mathbb{Z}\mathbb{A}_\infty$.*
- (2) *If Q has no left infinite path, then $\Gamma_{\text{rep}^+(Q)}$ has two regular components of which r are of shape $\mathbb{N}^-\mathbb{A}_\infty$ and $(2-r)$ are of shape $\mathbb{Z}\mathbb{A}_\infty$.*
- (3) *If Q has no right infinite path, then $\Gamma_{\text{rep}^+(Q)}$ has two regular components of which l are of shape $\mathbb{N}\mathbb{A}_\infty$ and $(2-l)$ are of shape $\mathbb{Z}\mathbb{A}_\infty$.*
- (4) *If Q has a left infinite maximal path and a right infinite maximal path, then $\Gamma_{\text{rep}^+(Q)}$ has two regular components of which one is of shape $\mathbb{Z}\mathbb{A}_\infty$ and the other one is a finite wing.*

Proof. We may assume that Q is canonical. It is easy to see that Q^+ is either empty or connected. Hence $\Gamma_{\text{rep}^+(Q)}$ has at most one preinjective component. Moreover, by Proposition 5.15(3), $\Gamma_{\text{rep}^+(Q)}$ has at most two regular components \mathcal{R} and \mathcal{L} , such that $M(p) \in \mathcal{R}$ for $p \in Q_R$, and $M(q) \in \mathcal{L}$ for $q \in Q_L$.

(1) Assume that Q is a double infinite path in which the arrows are all right-oriented. Then $Q_R = \emptyset$ and $Q_L = \{\varepsilon_x \mid x \in Q_0\}$. By Proposition 5.15, \mathcal{L} is the only regular component. Since $\varepsilon_i = \sigma_L^{-i}(\varepsilon_0)$, by Lemma 5.14, $\tau^i S_0 = S_i$, for $i \in \mathbb{Z}$. Hence, \mathcal{L} is stable. By Theorem 4.14(1), \mathcal{L} is of shape $\mathbb{Z}\mathbb{A}_\infty$.

(2) Suppose that Q has no left infinite path. Since $Q_0 = \mathbb{Z}$, there exist at most two right infinite maximal paths. Assume first that Q has no right infinite path, that is, it has no infinite path. Then Q can be viewed of the following form :

$$\cdots p_n q_n^{-1} \cdots q_1^{-1} p_0 q_0^{-1} p_{-1} \cdots p_{-m} q_{-m}^{-1} \cdots$$

where the p_n are the right-oriented maximal paths and the q_n are the left-oriented maximal paths. Thus Q_R contains a double infinite chain

$$\cdots \prec p_{-m} \prec \cdots \prec p_0 \prec \cdots \prec p_n \prec \cdots,$$

and hence $\sigma_R^i(p_0)$ is defined for all $i \in \mathbb{Z}$. By Lemma 5.13, $\tau^i M(p_0)$ is defined for all $i \in \mathbb{Z}$, that is, $M(p_0)$ is stable. Moreover, Q_L contains an infinite chain

$$\cdots \prec q_{-m} \prec \cdots \prec q_0 \prec \cdots \prec q_n \prec \cdots,$$

and thus $\sigma_L^{-i}(q_0)$ is defined for all $i \in \mathbb{Z}$. By Lemma 5.14, $\tau^i M(q_0)$ is defined for all $i \in \mathbb{Z}$. That is, $M(q_0)$ is stable. Therefore, $\Gamma_{\text{rep}^+(Q)}$ has two stable regular components \mathcal{R} and \mathcal{L} . By Theorem 4.14(1), they both are of shape $\mathbb{Z}\mathbb{A}_\infty$.

Assume next that Q has exactly one right infinite maximal path q_0 . We may assume that q_0 is left-oriented starting in x . Since q_0 is the unique infinite maximal path, Q can be viewed of the following form :

$$\cdots q_n^{-1} p_n \cdots q_1^{-1} p_1 q_0^{-1},$$

where the q_n are the left-oriented maximal paths and the p_n are the right-oriented maximal paths. Then Q_L contains a right infinite chain

$$q_0 \prec q_1 \prec \cdots \prec q_n \prec \cdots,$$

and thus, $\sigma_L^{-i}(q_0)$ is defined for all $i \geq 0$. By Lemma 5.14(1), $\tau^i M(q_0)$ is defined for all $i \geq 0$. That is, $M(q_0)$ is left stable. Hence \mathcal{L} is a left stable regular component. Since $M(q_0)$ is infinite dimensional, by Theorem 4.14(2), \mathcal{L} is of shape $\mathbb{N}^- \mathbb{A}_\infty$. On the other hand, Q_R has a double infinite chain

$$\cdots \prec \varepsilon_{x-i} \prec \cdots \prec \varepsilon_{x-1} \prec p_1 \prec \cdots \prec p_n \prec \cdots.$$

Making use of Lemma 5.13 again, we see that \mathcal{R} is stable, which is of shape $\mathbb{Z} \mathbb{A}_\infty$ by Theorem 4.14.

Assume finally that Q has two right infinite maximal paths p and q . We may assume that p is right-oriented starting in x , and q is left-oriented starting in y . Then Q_R has a left infinite chain

$$\cdots \prec \varepsilon_{y-i} \prec \cdots \prec \varepsilon_{y-1} \prec p.$$

In view of Lemma 5.13(1), we deduce that $M(p)$ is left stable. Hence \mathcal{R} is a left stable regular component. Since $M(p)$ is infinite dimensional, \mathcal{R} is of shape $\mathbb{N}^- \mathbb{A}_\infty$. Moreover, since Q_L contains a right infinite chain

$$q \prec \varepsilon_{x+1} \prec \cdots \prec \varepsilon_{x+i} \prec \cdots,$$

by Lemma 5.14(1), $M(q)$ is left stable. Hence, \mathcal{L} is a left stable regular component. Since $M(q)$ is infinite dimensional, \mathcal{L} is of shape $\mathbb{N}^- \mathbb{A}_\infty$.

(3) Suppose that Q has no right infinite path. Using an argument dual to that for proving Statement (2), we may show that if Q has exactly one left infinite maximal path which is assumed to be right-oriented, then \mathcal{L} is a regular component of shape $\mathbb{Z} \mathbb{A}_\infty$ and \mathcal{R} is a regular component of shape $\mathbb{N} \mathbb{A}_\infty$; and if Q has two left infinite maximal paths, then \mathcal{R} and \mathcal{L} are two regular components of shape $\mathbb{N} \mathbb{A}_\infty$.

(4) Suppose that Q has a left infinite maximal path p and a right infinite maximal path q . Then p, q are either both right-oriented or both left-oriented. We need only to consider the case where p is right-oriented with $e(p) = x$, while q is right-oriented with $s(q) = y$. Then Q_L has a double infinite chain

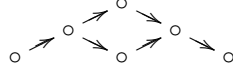
$$\cdots \prec \varepsilon_{x-i} \prec \cdots \prec \varepsilon_{x-1} \prec \varepsilon_{y+1} \prec \cdots \prec \varepsilon_{y+j} \prec \cdots.$$

Therefore, \mathcal{L} is stable of shape $\mathbb{Z} \mathbb{A}_\infty$. On the other hand, $q \in Q_R$. If z is a vertex with $z < x$ or $z > y$, then z is a middle point of a right-oriented path. By Lemma 5.4(1), $z \neq s(v)$ for any $v \in Q_R$. Hence, Q_R is finite. By Proposition 5.15(1), \mathcal{R} has a finite τ -orbit. By Theorem 4.14(4), \mathcal{R} is a finite wing. The proof of the theorem is completed.

EXAMPLE. Reconsider the canonical quiver Q of type \mathbb{A}_∞^∞ as follows :

$$\cdots \longleftarrow 0 \longleftarrow 1 \longleftarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6 \longleftarrow 7 \longleftarrow \cdots$$

As seen before, $Q_L = \{p_\infty, p_{4,3}, \varepsilon_5\}$. Therefore, the regular component containing the τ -orbit \mathcal{O}_L is a wing as follows:



On the other hand, since both σ and σ^- are defined everywhere in Q_R , the regular component containing \mathcal{O}_R is of shape $\mathbb{Z}\mathbb{A}_\infty$.

Finally, it comes to the point for us to study the \mathbb{D}_∞ -case. Recall from Lemma 5.8 that the indecomposable representations in $\text{rep}^+(Q)$ are uniquely determined by their dimension vector.

5.18. NOTATION. Let Q be an infinite Dynkin quiver of type \mathbb{D}_∞ .

- (1) For integers $i \geq 0$ and $j \geq 1$, denote by $N_{i,j}$ the finite dimensional indecomposable representation in $\text{rep}^+(Q)$ with dimension vector indicated as follows :

$$1 - \overbrace{2 - 2 - \dots - 2}^{i \text{ times}} - \overbrace{1 - \dots - 1}^{j \text{ times}} - 0 - 0 - \dots$$

$$\begin{array}{c} | \\ 1 \end{array}$$

- (2) In case Q has infinite paths, for each $i \geq 0$, denote by $N_{i,\infty}$ the sincere indecomposable representation in $\text{rep}(Q)$ with dimension vector indicated as follows :

$$1 - \overbrace{2 - 2 - \dots - 2}^{i \text{ times}} - 1 - \dots - 1 - \dots$$

$$\begin{array}{c} | \\ 1 \end{array}$$

The following result describes the indecomposable representations in $\text{rep}^+(Q)$.

5.19. PROPOSITION. *Let Q be a quiver of type \mathbb{D}_∞ . If M is an indecomposable representation in $\text{rep}^+(Q)$, then*

- (1) $M \cong M(w)$ with w a string having no left infinite path, or
- (2) $M \cong N_{i,j}$ with $i \geq 0$ and $j \geq 1$, or
- (3) $M \cong N_{i,\infty}$ with $i \geq 0$, and this occurs if and only if Q has right infinite paths.

Proof. We may assume that Q is canonical. Let M be an indecomposable representation in $\text{rep}^+(Q)$. If one of the vertices $0, 1$ is not in $\text{supp}M$, then M is an indecomposable representation of a quiver of type \mathbb{A}_∞ . By Proposition 5.9, $M = M(w)$, where w is a string without left infinite paths.

Suppose that $0, 1 \in \text{supp}M$. If M is finite dimensional, then it is an indecomposable representation of $\text{supp}M$. Since $\text{supp}M$ is of type \mathbb{D}_n , it is well known that $M \cong N_{i,j}$ for some $i \geq 0$ and $j \geq 1$; see, for example, [1, p. 299]. Assume that M is infinite dimensional. Since M is indecomposable, $\text{supp}M = Q$. By Corollary 1.7, Q contains a right infinite path which we assume starts in a vertex $a \geq 3$. By Theorem 1.12(1), M is projective restricted to a co-finite successor-closed subquiver Σ of Q . By the dual of Lemma 1.10, we may assume that the vertices x with $x \leq a$ are not in Σ . Let b be maximal such that b lies in the top-support of M_x , and let Ω be the full subquiver of Q generated by the vertices x with $0 \leq x \leq b$. Then Ω contains the top-support of M_x and is predecessor-closed in Q since $b > a$. By Theorem 1.13(1), M_Ω is an indecomposable sincere representation of Ω . Since Ω is of type \mathbb{D}_{b+1} , it is well known that there exists some r with $0 \leq r \leq b-2$ such that $\dim M(x) = 2$ for any $2 \leq x < 2+r$ and $\dim M(y) = 1$ if $y < 2$ or $2+r \leq y \leq b$.

In particular, $\dim M(b) = 1$. Let c be an arbitrary vertex with $c > b$. Applying the same argument to the full subquiver of Q generated by the vertices x with $0 \leq x \leq c$, we see that $\dim M(c) = 1$. This shows that $M \cong N_{r,\infty}$. The proof of the proposition is completed.

REMARK. The preceding result says particularly that if M is an indecomposable representation in $\text{rep}^+(Q)$, then $\dim M(x) \leq 2$ for all $x \in Q_0$.

As for the two other types, the study of quasi-simple representations is essential in the description of the Auslander-Reiten components of $\text{rep}^+(Q)$.

5.20. LEMMA. *Let Q be a quiver of type \mathbb{D}_∞ . If Γ is a regular component of $\Gamma_{\text{rep}^+(Q)}$, then every vertex in Q lies in the support of at most two quasi-simple representations in Γ .*

Proof. Fix $x \in Q_0$. Let Γ be a regular component of $\Gamma_{\text{rep}^+(Q)}$ with a sectional path $L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_1$, where L_1 is quasi-simple. In view of the shape of Γ described in Theorem 4.14, we see that $\tau^i L_1 \in \Gamma$, for $i = 1, \dots, n-1$. Now an easy induction on n shows that $\dim L_n(x) = \sum_{i=0}^{n-1} \dim \tau^i L_1(x)$.

Suppose that Γ contains some distinct quasi-simple representations N_1, N_2 , and N_3 such that $N_i(x) \neq 0$, for $i = 1, 2, 3$. With no loss of generality, we may assume that $N_3 = \tau^s N_1$ and $N_2 = \tau^r N_1$ with $0 < r < s$. Then Γ contains a sectional path $M_{s+1} \longrightarrow M_s \longrightarrow \cdots \longrightarrow M_1 = N_1$. This yields

$$\dim M_{s+1}(x) = \sum_{i=0}^s \dim \tau^i N_1(x) \geq \dim N_1(x) + \dim N_2(x) + \dim N_3(x) \geq 3,$$

which is contrary to Proposition 5.19. The proof of the lemma is completed.

5.21. LEMMA. *Let Q be a canonical quiver of type \mathbb{D}_∞ , and let w be a finite string with $3 \leq s(w) \leq e(w)$. If there exist arrows α, β such that $\beta w \alpha$ or $\beta^{-1} w \alpha^{-1}$ is a string, then $M(w)$ is a regular representation.*

Proof. Write $a = s(w)$ and $b = e(w)$. First, suppose that Q has a string $\beta w \alpha$ with α, β being arrows. Then α is the arrow $(a-1) \rightarrow a$ and β is the arrow $b \rightarrow (b+1)$. Applying Lemma 2.12(1) to the unique infinite acyclic walk in Q starting with β , we see that either $M(w)$ is left stable or $\tau^n M(w)$ is pseudo-projective for some integer $n \geq 0$. In particular, $M(w)$ is not preprojective.

For each integer $j \geq 1$, consider the indecomposable representation $N_{b-1,j}$ as defined in Notation 5.18(1). We may assume that $N_{b-1,j}(x) = k^2$ for $2 \leq x \leq b$, $N_{b-1,j}(b+1) = k$, and $N_{b-1,j}(\gamma) = \mathbf{1}$ for each arrow $\gamma : x \rightarrow y$ with $2 \leq x, y \leq b$. Since $N_{b-1,j}$ is indecomposable, the map $N_{b-1,j}(\beta)$ is surjective with a non-zero kernel $\varphi : k \rightarrow k^2$. On the other hand, $M(w)(x) = k$ for each vertex x appearing in w , and $M(w)(\gamma) = \mathbf{1}$ for each arrow γ such that γ or γ^{-1} is an edge in w . For $x \in Q_0$, we set $f(x) = \varphi$ if w passes through x ; and otherwise, $f(x) = 0$. By definition, $N_{b-1,j}(\beta) f(b) = 0 = f(b+1) M(w)(\beta)$, and since $M(w)(a-1) = 0$, we have $N_{b-1,j}(\alpha) f(a-1) = 0 = f(a) M(w)(\alpha)$. Moreover, if $\gamma : x \rightarrow y$ is an arrow such that γ or γ^{-1} is an edge in w , then $a \leq x, y \leq b$, and in this case, we have $N_{b-1,j}(\gamma) f(x) = \varphi = f(y) M(w)(\gamma)$. This verifies that $f = \{f(x) \mid x \in Q_0\}$ is a non-zero morphism in $\text{rep}^+(Q)$ from $M(w)$ to $N_{b-1,j}$. Since the $N_{b-1,j}$ with $j \geq 1$ are pairwise non-isomorphic, by Proposition 4.10(2), $M(w)$ is not preinjective. That is, $M(w)$ is regular.

Next, suppose that Q has a string $\beta^{-1}w\alpha^{-1}$ with α, β being arrows. Then we deduce from Lemma 2.13(1) that $M(w)$ is not preinjective. Moreover, in a dual manner, we can show that there exists a non-zero morphism from $N_{b-1,j}$ to $M(w)$ for each $j \geq 1$. Therefore, $M(w)$ is not preprojective by Proposition 4.10(1). That is, $M(w)$ is regular. The proof of the lemma is completed.

We are ready to describe the connected components of $\Gamma_{\text{rep}^+(Q)}$ in the \mathbb{D}_∞ -case. By Theorems 4.6 and 4.7, we need only to concentrate on the regular components.

5.22. THEOREM. *Let Q be a quiver of type \mathbb{D}_∞ . Then $\Gamma_{\text{rep}^+(Q)}$ consists of the preprojective component, at most one preinjective component, and exactly one regular component which is of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty$, or $\mathbb{N}^-\mathbb{A}_\infty$ in case Q has no infinite path, has left infinite paths, or has right infinite paths, respectively.*

Proof. Observing that Q^+ is empty or connected, we deduce easily the first two parts of the statement from Theorems 4.6 and 4.7. For proving the last part, assume that Q is canonical. If Q has a vertex x with $x > 2$ which is a middle point of some path, then S_x is regular by Lemma 5.21. Otherwise, every arrow α not attached to 2 is a maximal path, and by Lemma 5.21 again, $M(\alpha)$ is regular. This shows that $\Gamma_{\text{rep}^+(Q)}$ has at least one regular component Γ . If Q has no infinite path, then Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$ by Corollary 4.16. If Q has left infinite paths, then every left infinite acyclic walk in Q is an almost-path, and hence, Γ is of shape $\mathbb{N}\mathbb{A}_\infty$ by Theorem 4.17(2). If Q has right infinite paths, then every right infinite acyclic walk in Q is an almost-path, and therefore, Γ is of shape $\mathbb{N}^-\mathbb{A}_\infty$ by Theorem 4.17(1).

It remains to show that Γ is the only regular component of $\Gamma_{\text{rep}^+(Q)}$. For this purpose, we fix a vertex $a > 2$ in such a way that a is a source vertex if Q has no infinite path, and otherwise, a is a middle point of an infinite path. Denote by Σ the full subquiver of Q generated by the vertices $x \geq a$. Observe that Σ is a quiver of type \mathbb{A}_∞ , which will become canonical if one replaces x by $x - a$. Let Σ_R and Σ_L be the sets of paths in Σ as defined in Notation 5.3, each of them is equipped with a source-translation written as σ_R for Σ_R and σ_L for Σ_L .

Consider first the case where Q has no left infinite path. Then Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{N}^-\mathbb{A}_\infty$. In particular, Γ is left stable. By Lemma 3.6, all but at most one quasi-simple representations in Γ are finite dimensional. By Lemma 5.20, Γ contains a finite dimensional quasi-simple representation M such that $\text{supp } \tau^i M \subseteq \Sigma \setminus \{a\}$ for all $i \geq 0$. In particular, the $\tau^i M$ are finite dimensional representations in $\text{rep}^+(\Sigma)$. For each $i \geq 0$, consider the almost split sequence

$$\eta_i : \quad 0 \longrightarrow \tau^{i+1}M \longrightarrow E_i \longrightarrow \tau^i M \longrightarrow 0$$

in $\text{rep}^+(Q)$. We claim that η_i is an almost split sequence in $\text{rep}^+(\Sigma)$. Indeed, assume that Q has a right infinite path. Then Σ is a right infinite path with a being the source vertex. For $x, y \geq a$, we denote by $p_{x,y}$ the path in Σ from x to y . Then $M = M(p_{r,s})$ for some $s \geq r > a$. In view of Lemma 2.5, we see that $\tau^i M = M(p_{r+i,s+i})$, and in particular, η_i lies in $\text{rep}^+(\Sigma)$ for all $i \geq 0$. Hence, our claim follows. Assume that Q has no infinite path. Then a is a source vertex in Q . In particular, Σ contains the predecessors of the successors of the vertices in $\text{supp } \tau^i M$, for all $i \geq 0$. By Proposition 2.16(1), η_i is an almost split sequence in $\text{rep}^+(\Sigma)$, for all $i \geq 0$. This establishes our claim. In particular, $\tau_\Sigma^i M = \tau^i M$, for all $i \geq 0$. Since E_0 is indecomposable in $\text{rep}^+(\Sigma)$, by Proposition 5.12, there exists a double-hook (q, β, p) in Σ such that $M = M(p)$. By Lemma 5.6, either $p, q \in \Sigma_R$

or $p, q \in \Sigma_L$. Since M is left stable in $\Gamma_{\text{rep}^+(\Sigma)}$, by Proposition 5.15, the second case occurs. Applying Lemma 5.14(1), we have

$$M(\sigma_L^{-i}(p)) = \tau_\Sigma^i M(p) = \tau_\Sigma^i M = \tau^i M \in \Gamma,$$

for all $i \geq 0$. Similarly, if Γ' is another regular component of $\Gamma_{\text{rep}^+(Q)}$, then there exists some $p' \in \Sigma_L$ such that $M(\sigma_L^{-i}(p')) \in \Gamma'$ for all $i \geq 0$. Since p, p' both lie in Σ_L , there exists some $t \in \mathbb{Z}$ such that $p' = \sigma_L^t(p)$. As a consequence, Γ intersects Γ' , and hence $\Gamma = \Gamma'$.

Consider now the case where Q has left infinite paths. Then, Γ is of shape $\mathbb{N}\mathbb{A}_\infty$. In particular, Γ contains pseudo-projective but no infinite dimensional representations. On the other hand, Q has no right infinite path. By the dual of what we have just proved, $\Gamma_{\text{rep}^-(Q)}$ has a unique regular component \mathcal{C} which contains infinite dimensional representations. By Proposition 4.12(2), Γ is obtained from \mathcal{C} by deleting the infinite dimensional representations. Thus Γ is the unique regular component of $\Gamma_{\text{rep}^+(Q)}$. The proof of the theorem is completed.

To sum up, we have obtained the following result; compare [37, (3.10)].

5.23. COROLLARY. *If Q is an infinite Dynkin quiver, then $\Gamma_{\text{rep}^+(Q)}$ has at most four connected components and at most two are regular components.*

6. NUMBER OF REGULAR COMPONENTS

As seen in the preceding section, the Auslander-Reiten quiver $\Gamma_{\text{rep}^+(Q)}$ of $\text{rep}^+(Q)$ has at most two regular components in the infinite Dynkin-case. To the contrary, the main objective of this section is to show that $\Gamma_{\text{rep}^+(Q)}$ has always infinitely many regular components provided that Q is not of finite or infinite Dynkin type. The following result is essential for this purpose.

6.1. LEMMA. *Let Σ be a finite subquiver of Q . If M is a regular representation in $\text{rep}(\Sigma)$, then it is a regular representation in $\text{rep}^+(Q)$.*

Proof. Let M be a regular representation in $\Gamma_{\text{rep}(\Sigma)}$. It is well known that the regular components of $\Gamma_{\text{rep}(\Sigma)}$ are stable tubes or of shape $\mathbb{Z}\mathbb{A}_\infty$. Thus $\text{rep}(\Sigma)$ admits an infinite chain of irreducible epimorphisms

$$\cdots \longrightarrow M_i \xrightarrow{f_i} M_{i-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M$$

and an infinite chain of irreducible monomorphisms

$$M \xrightarrow{g_1} N_1 \longrightarrow \cdots \longrightarrow N_{i-1} \xrightarrow{g_i} N_i \longrightarrow \cdots$$

In particular, $\text{Hom}(M_i, M) \neq 0$ and $\text{Hom}(M, N_i) \neq 0$, for every $i \geq 1$. By the first two statements of Proposition 4.10, M is regular in $\text{rep}^+(Q)$. The proof of the lemma is completed.

6.2. LEMMA. *If Q is infinite, then every regular component of $\Gamma_{\text{rep}^+(Q)}$ has at most finitely many representations supported by any given finite full subquiver of Q .*

Proof. Let Q be infinite with a finite full subquiver Σ . Assume that Γ is a regular component of $\Gamma_{\text{rep}^+(Q)}$ containing infinitely many representations M_j with $j \geq 0$ such that $\text{supp} M_j \subseteq \Sigma$. Since Σ is finite, we may assume that $\text{supp} M_j = \Sigma$ for all $j \geq 0$. Setting n_j to be the quasi-length of M_j , we deduce from Proposition 2.14(2)

that the n_j are pairwise distinct. In particular, we may assume that $n_j > 0$ for all $j > 0$. By Theorem 4.14, Γ is of shape $\mathbb{Z}\mathbb{A}_\infty$, $\mathbb{N}\mathbb{A}_\infty$ or $\mathbb{N}^-\mathbb{A}_\infty$. Thus, for each $j > 0$, there exists in $\text{rep}^+(Q)$ a chain of irreducible epimorphisms of length $n_j - 1$ from M_j to a quasi-simple representation N_j , and a chain of irreducible monomorphisms of length $n_j - 1$ from $\tau^{n_j-1}N_j$ to M_j . As a consequence, the N_j and the $\tau^{n_j-1}N_j$ with $j > 0$ are all supported by Σ . Since the n_j are pairwise distinct, the set $\mathcal{Y} = \{N_j, \tau^{n_j-1}N_j \mid j \geq 0\}$ is infinite. Since Σ is finite, \mathcal{Y} contains infinitely many quasi-simple representations having the same support. This is contrary to Proposition 2.14(2). The proof of the lemma is completed.

Recall that a trivially valued translation quiver is called a *stable tube* of rank $n (> 0)$ if it is of shape $\mathbb{Z}\mathbb{A}_\infty / \langle \tau^n \rangle$, and a stable tube of rank one is called a *homogeneous tube*. The following result seems to be well known. However, to the best of our knowledge, it is not explicitly stated anywhere. For this reason, we include a proof which is suggested by Kerner.

6.3. PROPOSITION. *Let Q be a finite connected quiver without oriented cycles.*

- (1) *If Q is of Euclidean type, then $\Gamma_{\text{rep}(Q)}$ has infinitely many homogeneous tubes.*
- (2) *If Q is not of Dynkin type, then $\Gamma_{\text{rep}(Q)}$ has infinitely many regular components.*

Proof. (1) Let Q be of Euclidean type. The regular components of $\Gamma_{\text{rep}(Q)}$ are pairwise orthogonal stable tubes; see [16]. Assume that Q has only two vertices a, b . Then Q is the Kronecker quiver K with exactly two arrows α, β from a to b . Note that the regular components of $\Gamma_{\text{rep}(K)}$ are all homogeneous tubes; see [16]. Denote by \mathcal{P} the set of monic irreducible polynomials over k , which is known to be infinite. For each $p \in \mathcal{P}$, define $M_p \in \text{rep}(K)$ by setting $M_p(a) = M_p(b) = k[x]/\langle p \rangle$, $M_p(\alpha) = \mathbf{1}$, and $M_p(\beta)$ to be the multiplication by the class $\bar{x} \in k[x]/\langle p \rangle$. Having a dimension vector of the form (t, t) , the representation M_p is regular; see [16], and quasi-simple since $\text{End}(M_p) \cong K[x]/\langle p \rangle$. Moreover, the M_p with $p \in \mathcal{P}$ are pairwise orthogonal, and hence lie in pairwise different components of $\Gamma_{\text{rep}(K)}$.

Assume now that Q has $n (> 2)$ vertices and Statement (1) holds for any Euclidean quiver of $n - 1$ vertices. Then $\Gamma_{\text{rep}(Q)}$ has at least one stable tube \mathcal{T} of rank $r > 1$; see [16, Section 6]. Choose a quasi-simple representation S in \mathcal{T} . The perpendicular category S^\perp , that is the full additive subcategory of $\text{rep}(Q)$ generated by the representations L with $\text{Hom}(S, L) = \text{Ext}^1(S, L) = 0$, is equivalent to $\text{rep}(Q')$, where Q' is an Euclidean quiver of $n - 1$ vertices; see [20, (10.1)]. Thus, the Auslander-Reiten quiver of S^\perp has infinitely many homogenous quasi-simple representations S_i with $i \geq 1$. Since $\text{Ext}^1(S_i, S_i) \neq 0$, we see that S_i lies in a stable tube \mathcal{T}_i of $\Gamma_{\text{rep}(Q)}$. If $\mathcal{T}_i = \mathcal{T}$ for some i then, since $\text{End}(S_i)$ is divisible, the quasi-length of S_i is at most r . Thus we may assume $\mathcal{T}_i \neq \mathcal{T}$, for all $i \geq 1$. Consider, for each $i \geq 1$, an almost split sequence

$$\eta_i : 0 \longrightarrow S_i \longrightarrow E_i \xrightarrow{g_i} S_i \longrightarrow 0$$

in S^\perp . Let $f : L \rightarrow S_i$ be a non-retraction morphism in $\text{rep}(Q)$ with $L \in \Gamma_{\text{rep}(Q)}$. Then L lies in \mathcal{T}_i or L is preprojective. In the first case, $L \in S^\perp$ since $\mathcal{T}_i \neq \mathcal{T}$, and hence, f factors through g_i . In the second case, consider a pull-back diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_i & \longrightarrow & E & \longrightarrow & L \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow f \\ 0 & \longrightarrow & S_i & \longrightarrow & E_i & \xrightarrow{g_i} & S_i \longrightarrow 0 \end{array}$$

in $\text{rep}(Q)$. Since $\text{Ext}_{\text{rep}(Q)}^1(L, S_i) \cong D\text{Hom}_{\text{rep}(Q)}(S_i, \tau L) = 0$, the upper row splits. Thus f factors through g_i . This shows that η_i is an almost split sequence in $\text{rep}(Q)$. Consequently, the \mathcal{T}_i with $i \geq 1$ are all homogenous tubes of $\Gamma_{\text{rep}(Q)}$.

(2) Let Q be of non-Dynkin type. By Statement (1), we may assume that Q is wild. Then every regular component of $\Gamma_{\text{rep}^+(Q)}$ is of shape $\mathbb{Z}\mathbb{A}_\infty$; see [40]. Consider first the case where Q has only two vertices. We may assume that Q consists of the above-mentioned Kronecker quiver K and possibly some extra arrows from a to b . Then $\text{rep}(K)$ is a full additive subcategory of $\text{rep}(Q)$ generated by the representations annihilated by the arrows other than α, β . By Lemma 6.1, the M_p with $p \in \mathcal{P}$ are regular representations in $\Gamma_{\text{rep}(Q)}$. Suppose that M_q is not quasi-simple in $\Gamma_{\text{rep}(Q)}$ for some $q \in \mathcal{P}$. Then $\text{rep}(Q)$ has an irreducible epimorphism $f: M_q \rightarrow N$. It is easy to verify that N is annihilated by the arrows other than α, β , and thus f is an irreducible epimorphism in $\text{rep}(K)$. This contradicts the fact that M_q is quasi-simple in $\Gamma_{\text{rep}(K)}$. Therefore, the M_p with $p \in \mathcal{P}$ are all quasi-simple in $\Gamma_{\text{rep}(Q)}$. Suppose on the contrary that $\Gamma_{\text{rep}(Q)}$ has only finitely many regular components. Being quasi-simple, the M_p with $p \in \mathcal{P}$ are contained in finitely many τ -orbits of $\Gamma_{\text{rep}(Q)}$. Therefore, \mathcal{P} contains infinitely many p_i with $i \geq 1$ such that the M_{p_i} lie in the the same τ -orbit of $\Gamma_{\text{rep}(Q)}$. Write $M_i = M_{p_i}$, for all $i \geq 1$. Since M_1 is regular, there exists some $s > 0$ such that $\text{Hom}(M_1, \tau^i M_1) \neq 0$ for every $i \geq s$; see [28, (1.3)]. In particular, $M_j = \tau^{n_j} M_1$ with $n_j \leq s$, for all $j \geq 1$. Then there exists some $r > 1$ such that $n_r < -s$, that is, $-n_r > s$. This yields

$$\text{Hom}(M_r, M_1) = \text{Hom}(\tau^{n_r} M_1, M_1) \cong \text{Hom}(M_1, \tau^{-n_r} M_1) \neq 0,$$

contrary to the fact that the M_p are pairwise orthogonal. Assume now that Q has $n (> 2)$ vertices and Statement (2) holds for any non-Dynkin quiver of $n-1$ vertices. Let Q' be a non-Dynkin connected full subquiver of Q with $n-1$ vertices. Then, $\Gamma_{\text{rep}(Q')}$ has infinitely many regular representations N_i with $i \geq 1$. By Lemma 6.1, the N_i are regular representations in $\Gamma_{\text{rep}(Q)}$. Since the N_i are not sincere, they are distributed in infinitely many regular components of $\Gamma_{\text{rep}(Q)}$; see [28, (1.3)]. The proof of the proposition is completed.

6.4. THEOREM. *If Q is a connected strongly locally finite quiver, then $\Gamma_{\text{rep}^+(Q)}$ has only finitely many regular components if and only if Q is of finite or infinite Dynkin type.*

Proof. Let Q be a connected strongly locally finite quiver. If Q is of finite Dynkin type, then $\Gamma_{\text{rep}^+(Q)}$ has no regular component. If Q is of infinite Dynkin type, then $\Gamma_{\text{rep}^+(Q)}$ has at most two regular components by Corollary 5.23. For proving the necessity, by Proposition 6.3, we only need to consider the case where Q is infinite but not of infinite Dynkin type. Then Q contains a connected finite full subquiver Σ of non-Dynkin type. By Proposition 6.3, $\Gamma_{\text{rep}^+(\Sigma)}$ contains infinitely many regular representations M_i with $i \geq 1$. By Lemma 6.1, the M_i are regular representations in $\Gamma_{\text{rep}^+(Q)}$. By Lemma 6.2, they are distributed in infinitely many regular components of $\Gamma_{\text{rep}^+(Q)}$. The proof of the theorem is completed.

To conclude this section, we shall show that any of the four types of regular components may appear infinitely many times. For this purpose, we need the following lemma.

6.5. LEMMA. *Suppose that Q is infinite and connected. Let M be a representation in $\Gamma_{\text{rep}^+(Q)}$ such that $\text{supp}M \subseteq \text{supp}(\tau^n M)$ for some $n > 0$. If Q has left infinite paths, then $\tau^m M$ is pseudo-projective for some $m \geq n$.*

Proof. Let p be a left infinite path in Q . By Proposition 3.6, $\tau^n M$ is finite dimensional. Since Q is connected, there exists a right infinite acyclic walk w which intersects $\text{supp}(\tau^n M)$ only at $s(w)$ and ends with the inverse of a left infinite subpath of p . In particular, all but finitely many edges in w are inverses of arrows. Assume that w starts with the inverse of an arrow, then w^{-1} is an infinite acyclic walk which ends with an arrow and intersects $\text{supp}(\tau^n M)$ only at $e(w^{-1})$. By Lemma 2.13(2), $\text{supp}M$ contains some vertex lying in w^{-1} but different from $e(w^{-1}) = S(w)$, which is absurd since $\text{supp}(\tau^n M) \supseteq \text{supp}M$. Thus, w starts with an arrow. By Lemma 2.12(2), $\text{DTr}^m M$ is infinite dimensional for some $m > n$, that is, $\tau^{m-1} M$ is a pseudo-projective representation in Γ . The proof of the lemma is completed.

6.6. THEOREM. *Let Q be a connected strongly locally finite quiver which is infinite but not of infinite Dynkin type.*

- (1) *If Q has no infinite path, then $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components of shape $\mathbb{Z}\mathbb{A}_\infty$.*
- (2) *If Q has left infinite but no right infinite paths, then $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components of shape $\mathbb{N}\mathbb{A}_\infty$.*
- (3) *If Q has right infinite but no left infinite paths, then $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components of shape $\mathbb{N}^-\mathbb{A}_\infty$.*
- (4) *If Q has both left infinite paths and right infinite paths, then $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components of wing type.*

Proof. Statement (1) is an immediate consequence of Theorem 6.4 and Corollary 4.16. For proving the rest of the theorem, let Σ be a wild finite full subquiver of Q . Then $\Gamma_{\text{rep}(\Sigma)}$ contains infinitely many regular representations M_i with $i \geq 1$ such that M_i , $\tau_\Sigma M_i$, and $\tau_\Sigma^- M_i$ are all sincere representations of Σ ; see [28, (1.3)]. By Lemma 6.1, each M_i lies in a regular component Γ_i of $\Gamma_{\text{rep}^+(Q)}$. By Lemma 6.2, the Γ_i with $i \geq 1$ are pairwise distinct.

Assume that Q has left infinite path. We claim that none of the M_i is left stable in $\Gamma_{\text{rep}^+(Q)}$. Indeed, let $i \geq 1$ be such that $\tau M_i \in \Gamma_i$. Then τM_i is finite dimensional. Choose Ω to be a finite connected full subquiver of Q containing the support of $M_i \oplus \tau M_i$. Then $\tau_\Omega M_i = \tau M_i$. Since $\Sigma = \text{supp}M_i \subseteq \Omega$, the path algebra $k\Sigma$ is a quotient of the path algebra $k\Omega$. By Lemma 5.2 stated in [1, Chapter VIII], $\tau_\Sigma M_i$ is a sub-representation of $\tau_\Omega M_i$. This yields

$$\text{supp}M_i = \text{supp}(\tau_\Sigma M_i) \subseteq \text{supp}(\tau_\Omega M_i) = \text{supp}(\tau M_i).$$

By Lemma 6.5(2), $\text{DTr}^r M_i$ is infinite dimensional in $\text{rep}(Q)$ for some $r > 0$. That is, M_i is not left stable in $\Gamma_{\text{rep}^+(Q)}$. This establishes our claim. Dually, we may show that if Q has right infinite paths, then none of the M_i is right stable in $\Gamma_{\text{rep}^+(Q)}$.

Now suppose that Q has left infinite but no right infinite paths. Then $\Gamma_{\text{rep}^+(Q)}$ has no infinite dimensional representation. By Theorem 4.14, the Γ_i are all of shape $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{N}\mathbb{A}_\infty$. By the above claim, none of the Γ_i are left stable. Hence, the Γ_i with $i \geq 1$ are all of shape $\mathbb{N}\mathbb{A}_\infty$. This proves Statement (2). Dually, Statement (3) holds true. Finally, suppose that Q has both left infinite paths and right infinite paths. As seen above, each of the M_i with $i \geq 1$ is neither left stable nor right

stable. By Theorem 4.14(4), the Γ_i are all finite wings. The proof of the theorem is completed.

7. THE BOUNDED DERIVED CATEGORIES

The main objective of this section is to study the Auslander-Reiten theory in $D^b(\text{rep}^+(Q))$, the derived category of the bounded complexes in $\text{rep}^+(Q)$. Making use of the previously obtained results for $\text{rep}^+(Q)$, we shall be able to give a complete description of its Auslander-Reiten components of $D^b(\text{rep}^+(Q))$.

We begin with an arbitrary hereditary abelian category \mathcal{H} . Let $D^b(\mathcal{H})$ stand for the derived category of the bounded complexes in \mathcal{H} . It is well known that $D^b(\mathcal{H})$ is a triangulated category whose translation functor is the *shift functor* denoted by $[1]$. If $f : X \rightarrow Y$ is a morphism in $D^b(\mathcal{H})$ and $i \in \mathbb{Z}$, then we call $X[i]$ and $f[i] : X[i] \rightarrow Y[i]$ the *shifts* by i of X and f , respectively. As usual, we shall regard \mathcal{H} as a full subcategory of $D^b(\mathcal{H})$ by identifying an object $X \in \mathcal{H}$ with the stalk complex $X[0]$. It is important to observe that each object in $D^b(\mathcal{H})$ is a finite direct sum of the stalk complexes $X[i]$, where $X \in \mathcal{H}$ and $i \in \mathbb{Z}$. Moreover, for $X, Y \in \mathcal{H}$, $\text{Hom}_{D^b(\mathcal{H})}(X[i], Y[j]) \neq 0$ only if $i \leq j \leq i + 1$; see [34, (3.1)]. Now, let

$$\Delta : \quad X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be an exact triangle in $D^b(\mathcal{H})$. One calls X the *starting term* and Z the *ending term* of Δ . We say that Δ is an *almost split triangle* if f is minimal left almost split and g is minimal right almost split; compare [21]. For various equivalent conditions, we refer the reader to [29, (2.6)]. One says that $D^b(\mathcal{H})$ has left (respectively, right) almost split triangles if every indecomposable object in $D^b(\mathcal{H})$ is the starting (respectively, ending) term of an almost split triangle, and that $D^b(\mathcal{H})$ has almost split triangles if it has left and right almost split triangles.

The following result tells us how the minimal almost split morphisms in $D^b(\mathcal{H})$ are related to those in \mathcal{H} .

7.1. LEMMA. *Let X, Y , and Z be objects in \mathcal{H} .*

- (1) *If $(f, \eta)^T : X \rightarrow Y \oplus Z[1]$ is a minimal left almost split morphism in $D^b(\mathcal{H})$, then $f : X \rightarrow Y$ is a minimal left almost split morphism in \mathcal{H} .*
- (2) *If $(g, \zeta) : Y \oplus Z[-1] \rightarrow X$ is a minimal right almost split morphism in $D^b(\mathcal{H})$, then $g : Y \rightarrow X$ is a minimal right almost split morphism in \mathcal{H} .*
- (3) *If $\xi : X \rightarrow Y[1]$ is an irreducible morphism in $D^b(\mathcal{H})$, then X is injective and Y is projective.*

Proof. Let $(f, \eta)^T : X \rightarrow Y \oplus Z[1]$ be a minimal left almost split morphism in $D^b(\mathcal{H})$. It is evident that $f : X \rightarrow Y$ is left minimal and is not a section. Suppose that $u : X \rightarrow M$ is a non-section morphism in \mathcal{H} . Then u factors through $(f, \eta)^T$ in $D^b(\mathcal{H})$. Since $\text{Hom}_{D^b(\mathcal{H})}(Z[1], M) = 0$, we see that u factors through f in \mathcal{H} . This proves Statement (1). In a dual manner, we can establish Statement (2). Finally, suppose that $\xi : X \rightarrow Y[1]$ is an irreducible morphism in $D^b(\mathcal{H})$. Let

$$0 \longrightarrow X \xrightarrow{u} M \xrightarrow{v} N \longrightarrow 0$$

be a short exact sequence in \mathcal{H} . Since $\text{Ext}_{\mathcal{H}}^2(N, Y) = 0$, there exists some morphism

$\delta : M \rightarrow Y[1]$ in $D^b(\mathcal{H})$ such that $\xi = \delta \circ u$. Since $\text{Hom}_{D^b(\mathcal{H})}(Y[1], M) = 0$, we see that δ is not a retraction in $D^b(\mathcal{H})$. Thus u is a section in $D^b(\mathcal{H})$, and hence a section in \mathcal{H} . Since \mathcal{H} is abelian, this shows that X is injective. Dually, one can show that Y is projective. The proof of the lemma is completed.

The following result relates the almost split triangles in $D^b(\mathcal{H})$ to the almost split sequences in \mathcal{H} . This was first established by Happel in the case where \mathcal{H} is the category of finite dimensional representations of a finite acyclic quiver; see [21, (5.4)]. Observe that our approach is very much different from Happel's.

7.2. THEOREM. *Let \mathcal{H} be a hereditary abelian category.*

- (1) *If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an almost split sequence in \mathcal{H} , then it induces an almost split triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $D^b(\mathcal{H})$.*
- (2) *If S is a simple object in \mathcal{H} with a projective cover P and an injective hull I , then $D^b(\mathcal{H})$ has an almost split triangle as follows:*

$$I \rightarrow (I/S) \oplus (\text{rad}P)[1] \rightarrow P[1] \rightarrow I[1].$$

- (3) *Every almost split triangle in $D^b(\mathcal{H})$ is a shift of an almost split triangle stated in the above two statements.*

Proof. (1) Let \mathcal{H} have an almost split sequence $\eta : 0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$. It induces an exact triangle $\Delta : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\eta} X[1]$ in $D^b(\mathcal{H})$. Since g is right minimal in \mathcal{H} , it is right minimal in $D^b(\mathcal{H})$. We claim that each non-zero non-retraction morphism $\zeta : M \rightarrow Z$ in $D^b(\mathcal{H})$ factors through g . Indeed, we may assume that $\zeta = (h, \xi) : M \oplus N[-1] \rightarrow Z$, where $M, N \in \mathcal{H}$. Then h is a non-retraction morphism in \mathcal{H} , and hence it factors through g in \mathcal{H} . On the other hand, since $\text{Hom}_{D^b(\mathcal{H})}(L[-1], X[1]) = 0$, we have $\eta\xi = 0$, and thus ξ factors through g in $D^b(\mathcal{H})$. This proves that g is minimal right almost split in $D^b(\mathcal{H})$. Hence, Δ is almost split; see [29, (2.6)].

(2) Let S be a simple object in \mathcal{H} with a projective cover $\varepsilon : P \rightarrow S$ and an injective hull $\iota : S \rightarrow I$. In view of Lemma 2.1, we see that P and I are strongly indecomposable. Setting $h = \iota\varepsilon$, we get an exact triangle

$$(*) \quad I[-1] \xrightarrow{f} M \xrightarrow{g} P \xrightarrow{h} I$$

in $D^b(\mathcal{H})$. Let $\mu : X \rightarrow P$ be a non-zero non-retraction morphism in $D^b(\mathcal{H})$. We may assume that $\mu = (u, \delta) : Y \oplus Z[-1] \rightarrow P$, where $Y, Z \in \mathcal{H}$. Then $u : Y \rightarrow P$ is a non-retraction morphism in \mathcal{H} . Thus $\varepsilon u = 0$, and hence, $hu = 0$. On the other hand, since $\text{Hom}_{D^b(\mathcal{H})}(Z[-1], I) \cong \text{Ext}_{\mathcal{H}}^1(Z, I) = 0$, we have $h\delta = 0$. This implies that $h\mu = 0$, and consequently, μ factors through g . Therefore, g is right almost split in $D^b(\mathcal{H})$. Since $I[-1]$ is strongly indecomposable, $(*)$ is an almost split triangle in $D^b(\mathcal{H})$; see [29, (2.6)]. Furthermore, we may assume that $M = N \oplus L[-1]$, where $N, L \in \mathcal{H}$. Write $f = (f_1, v[-1])^T$ and $g = (w, g_1)$, where $v : I \rightarrow L$ and $w : N \rightarrow P$ are morphisms in \mathcal{H} . By Lemma 7.1, v is minimal left almost split, and w is minimal right almost split. Hence, $N \cong \text{rad}P$ and $L \cong I/S$.

(3) Let $\Delta : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be an almost split triangle in $D^b(\mathcal{H})$. Up to a shift, we may assume that $X \in \mathcal{H}$. Since g is right minimal, we may assume that $Y = M \oplus N[1]$ with $M, N \in \mathcal{H}$. Write $f = (u, \zeta)^T : X \rightarrow M \oplus N[1]$ with

$u : X \rightarrow M$ a morphism in \mathcal{H} , and $g = (\xi, v) : M \oplus N[1] \rightarrow Z$. By Lemma 7.1(1), $u : X \rightarrow M$ is minimal left almost split in \mathcal{H} .

Consider first the case where X is not injective in \mathcal{H} . Then \mathcal{H} has a non-section monomorphism $X \rightarrow L$, which factors through u . In particular, u is a minimal left almost split monomorphism in \mathcal{H} . Then \mathcal{H} has an almost split sequence

$$\zeta : 0 \longrightarrow X \xrightarrow{u} M \xrightarrow{w} N \longrightarrow 0;$$

see, for example, [6, (2.13), (2.14)]. By Statement (1), $X \xrightarrow{u} M \xrightarrow{w} N \xrightarrow{\zeta} X[1]$ is an almost split triangle $D^b(\mathcal{H})$, which is isomorphic to Δ . In other words, Δ is of the form as stated in Statement (1).

Consider next the case where $X = I$, an injective object in \mathcal{H} . Then $u : I \rightarrow M$ is a minimal left almost split epimorphism in \mathcal{H} . Let $q : S \rightarrow I$ be the kernel of u . By Lemma 2.1(2), S is simple with q being its injective hull, and $M \cong I/S$. Suppose that $Z \in \mathcal{H}$. Then $v = 0$ and $\xi u = -v\zeta = 0$. Since $u : X \rightarrow M$ is an epimorphism in \mathcal{H} , we get $\xi = 0$, and hence $g = 0$. As a consequence, $h : Z \rightarrow I[1]$ is a section, which is impossible since $\text{Hom}_{D^b(\mathcal{H})}(I[1], Z) = 0$. This shows that $Z \notin \mathcal{H}$. Since $h \neq 0$ and Z is indecomposable, $Z = P[1]$ for some $P \in \mathcal{H}$. Then, $h = s[1]$ and $v = j[1]$, where $s : P \rightarrow I$ and $j : N \rightarrow P$ are morphisms in \mathcal{H} . Now

$$\Delta[-1] : I[-1] \xrightarrow{f[-1]} Y[-1] \xrightarrow{g[-1]} P \xrightarrow{s} I,$$

is an almost split triangle in $D^b(\mathcal{H})$, where $g[-1] = (\xi[-1], j) : M[-1] \oplus N \rightarrow P$. By Lemma 7.1(2), $j : N \rightarrow P$ is minimal right almost split in \mathcal{H} . If P is not projective, then we can show that $\Delta[-1]$ is isomorphic to an almost split triangle induced from an almost split sequence in \mathcal{H} ending with P . In particular, $I[-1]$ is isomorphic to an object in \mathcal{H} , which is absurd. Thus P is projective. Therefore, $j : N \rightarrow P$ is a minimal right almost split monomorphism. Let $\varepsilon : P \rightarrow T$ be the cokernel of j . By Lemma 2.1(1), T is simple with ε being its projective cover and $N \cong \text{rad}P$. Moreover, since $s \circ g[-1] = 0$ and $fs = 0$, we have $sj = 0$ and $us = 0$. This yields a factorization $s = qp\varepsilon$, where $p : T \rightarrow S$ is a non-zero morphism in \mathcal{H} . Since T, S are simple, p is an isomorphism. That is, Δ is of the form as stated in Statement (2). The proof of the theorem is completed.

Combining Theorem 7.2 and Corollary 2.2 yields immediately the following consequence. In the Hom-finite case, this is a result of Reiten and Van Den Bergh, which is stated without a complete proof in [39, (I.3.2)].

7.3. COROLLARY. *If \mathcal{H} is an arbitrary hereditary abelian category, then*

- (1) $D^b(\mathcal{H})$ has left almost split triangles if and only if \mathcal{H} is left Auslander-Reiten and the socle of any indecomposable injective object has a projective cover;
- (2) $D^b(\mathcal{H})$ has right almost split triangles if and only if \mathcal{H} is right Auslander-Reiten and the top of any indecomposable projective object has an injective hull.

From now on, we shall specialize our previous results to the case where \mathcal{H} is an abelian full subcategory of $\text{rep}(Q)$. First of all, combining Theorems 2.8 and 7.2, we get immediately the following description of certain almost split triangles in $D^b(\text{rep}(Q))$.

7.4. THEOREM. *Let Q be a strongly locally finite quiver, and let $M \in \text{rep}(Q)$ be indecomposable.*

- (1) If $M \in \text{rep}^+(Q)$ is not projective, then $D^b(\text{rep}(Q))$ has an almost split triangle $\text{DTr} M \longrightarrow N \longrightarrow M \longrightarrow (\text{DTr} M)[1]$.
- (2) If $M \in \text{rep}^-(Q)$ is not projective, then $D^b(\text{rep}(Q))$ has an almost split triangle $M \longrightarrow N \longrightarrow \text{Tr} D M \longrightarrow M[1]$.
- (3) If $x \in Q_0$, then $D^b(\text{rep}(Q))$ has an almost split triangle

$$I_x \longrightarrow I_x/S_x \oplus (\text{rad} P_x)[1] \longrightarrow P_x[1] \longrightarrow I_x[1].$$

The rest of the section is devoted to our main objective, that is, to study the Auslander-Reiten theory in $D^b(\text{rep}^+(Q))$. We begin with a complete description of its almost split triangles.

7.5. THEOREM. *Let Q be a strongly locally finite quiver, and let $M \in \text{rep}^+(Q)$ be indecomposable.*

- (1) *If M is neither projective nor pseudo-projective, then $D^b(\text{rep}^+(Q))$ has an almost split triangle $\text{DTr} M \longrightarrow N \longrightarrow M \longrightarrow (\text{DTr} M)[1]$.*
- (2) *If M is finite dimensional and not injective, then $D^b(\text{rep}^+(Q))$ has an almost split triangle $M \longrightarrow N \longrightarrow \text{Tr} D M \longrightarrow M[1]$.*
- (3) *If x is a vertex in Q^+ , then $D^b(\text{rep}^+(Q))$ has an almost split triangle*

$$I_x \longrightarrow I_x/S_x \oplus (\text{rad} P_x)[1] \longrightarrow P_x[1] \longrightarrow I_x[1].$$

- (4) *Every almost split triangle in $D^b(\text{rep}^+(Q))$ is a shift of an almost split triangle stated in the above three statements.*

Proof. The first three statements follow immediately from Theorems 2.8 and 7.2. Now consider an almost split triangle $\Delta : L \longrightarrow M \longrightarrow N \longrightarrow L[1]$ in $D^b(\text{rep}^+(Q))$. Up to a shift, we may assume that $L \in \text{rep}^+(Q)$. If Δ is induced from an almost split sequence in $\text{rep}^+(Q)$, then it is of the form stated in Statement (1) or (2). Otherwise, by Theorem 7.2, $N = P_x[1]$ for some $x \in Q_0$ and L is the injective hull of S_x in $\text{rep}^+(Q)$. By Proposition 1.16(3), $x \in Q^+$, and hence $L \cong I_x$. That is, Δ is of the form as stated in Statement (3). The proof of the theorem is completed.

Next, we want to describe the irreducible morphisms in $D^b(\text{rep}^+(Q))$. Being non-zero, they are of the form $f : M[i] \rightarrow N[i]$ or $\zeta : M[i] \rightarrow N[i+1]$, where $M, N \in \text{rep}^+(Q)$ and $i \in \mathbb{Z}$. It is evident that we need only to study the irreducible morphisms of the second kind.

7.6. LEMMA. *Let M, N be representations in $\text{rep}^+(Q)$. If M is indecomposable, then $D^b(\text{rep}^+(Q))$ has an irreducible morphism $\eta : M \rightarrow N[1]$ if and only if there exists some $x \in Q^+$ such that $M \cong I_x$ and N is a direct summand of $\text{rad} P_x$.*

Proof. Suppose that M is indecomposable. The sufficiency follows easily from Theorem 7.5(3). Let $\eta : M \rightarrow N[1]$ be an irreducible morphism in $D^b(\text{rep}^+(Q))$. We claim that M is finite dimensional. Indeed, by Lemma 7.1(3) and Proposition 1.16, N has some P_y with $y \in Q_0$ as a direct summand. Then $D^b(\text{rep}^+(Q))$ has an irreducible morphism $\zeta : M \rightarrow P_y[1]$; see [9, (3.2)]. By Theorem 7.4(3), $D^b(\text{rep}(Q))$ has an almost split triangle

$$I_y \longrightarrow I_y/S_y \oplus (\text{rad} P_y)[1] \xrightarrow{(\theta, q)} P_y[1] \longrightarrow I_y[1].$$

Since ζ is not a retraction in $D^b(\text{rep}(Q))$, we have $\zeta = \theta f + q\xi$, where $f : M \rightarrow I_y/S_y$ is a morphism in $\text{rep}(Q)$ and $\xi : M \rightarrow (\text{rad}P_y)[1]$ is a morphism in $D^b(\text{rep}(Q))$. The composition of θ and f is given by a pull-back diagram

$$\begin{array}{ccccccccc} \theta f : & 0 & \longrightarrow & P_y & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow f & & \\ \theta : & 0 & \longrightarrow & P_y & \longrightarrow & U & \longrightarrow & I_y/S_y & \longrightarrow & 0 \end{array}$$

in $\text{rep}(Q)$, where $L \in \text{rep}^+(Q)$. Let Σ be the successor-closed subquiver of Q generated by the vertices in $\text{supp}L$. Restricting the preceding pull-back diagram to Σ yields a pull-back diagram

$$\begin{array}{ccccccccc} \theta f : & 0 & \longrightarrow & P_y & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow f_\Sigma & & \\ \theta_\Sigma : & 0 & \longrightarrow & P_y & \longrightarrow & U_\Sigma & \longrightarrow & (I_y/S_y)_\Sigma & \longrightarrow & 0 \end{array}$$

in $\text{rep}^+(Q)$. That is, we have a factorization $\zeta = (\theta_\Sigma, q)(f_\Sigma, \xi)^T$ in $D^b(\text{rep}^+(Q))$. Since $(\theta_\Sigma, q) : (I_y/S_y)_\Sigma \oplus \text{rad}P_y[1] \rightarrow P_y[1]$ is clearly not a retraction, the morphism

$$\begin{pmatrix} f_\Sigma \\ \xi \end{pmatrix} : M \rightarrow (I_y/S_y)_\Sigma \oplus (\text{rad}P_y)[1]$$

is a section. Then, f_Σ is a section, and hence M is a direct summand of $(I_y/S_y)_\Sigma$. On the other hand, since Σ is top-finite by Lemma 1.6, $(I_y/S_y)_\Sigma$ is finite dimensional, and so is M . This establishes our claim. Then, by Proposition 1.16 and Lemma 7.1(3), we may assume that $M = I_x$ with $x \in Q^+$. By Theorem 7.5(3), $D^b(\text{rep}^+(Q))$ has an almost split triangle

$$I_x \longrightarrow I_x/S_x \oplus (\text{rad}P_x)[1] \longrightarrow P_x[1] \longrightarrow I_x[1].$$

Since $\eta : I_x \rightarrow N[1]$ is an irreducible morphism in $D^b(\text{rep}^+(Q))$, one has a retraction $(\xi, w) : I_x/S_x \oplus (\text{rad}P_x)[1] \rightarrow N[1]$. Since $\text{Hom}_{D^b(\text{rep}^+(Q))}(N[1], I_x/S_x) = 0$, we see that $w : (\text{rad}P_x)[1] \rightarrow N[1]$ is a retraction. As a consequence, N is a direct summand of $\text{rad}P_x$. The proof of the lemma is completed.

We are ready to describe the Auslander-quiver $\Gamma_{D^b(\text{rep}^+(Q))}$ of $D^b(\text{rep}^+(Q))$. Since $\text{rep}^+(Q)$ is hereditary and abelian, the vertices in $\Gamma_{D^b(\text{rep}^+(Q))}$ can be chosen to be the complexes of the form $M[i]$, and then the arrows are of the form $M[i] \rightarrow N[i]$ or $M[i] \rightarrow N[i+1]$, where $M, N \in \Gamma_{\text{rep}^+(Q)}$ and $i \in \mathbb{Z}$.

7.7. LEMMA. *Let Q be a strongly locally finite quiver.*

- (1) *If $\alpha : M \rightarrow N$ is an arrow in $\Gamma_{\text{rep}^+(Q)}$, then it is also an arrow in $\Gamma_{D^b(\text{rep}^+(Q))}$.*
- (2) *If $\beta : x \rightarrow y$ is an arrow in Q , where $x \in Q^+$, then it induces an arrow $[\beta] : I_x \rightarrow P_y[1]$ in $\Gamma_{D^b(\text{rep}^+(Q))}$.*
- (3) *Each arrow in $\Gamma_{D^b(\text{rep}^+(Q))}$ is a shift of an arrow stated in the above two statements. In particular, $\Gamma_{D^b(\text{rep}^+(Q))}$ has a symmetric valuation.*

Proof. First of all, if X is a representation in $\Gamma_{\text{rep}^+(Q)}$, then $\text{End}_{\text{rep}^+(Q)}(X)$ and $\text{End}_{D^b(\text{rep}^+(Q))}(X)$ have the same residue algebra which we denote by k_X .

- (1) Let $M, N \in \Gamma_{\text{rep}^+(Q)}$ such that the number of arrows from M to N in $\Gamma_{\text{rep}^+(Q)}$ is $d_{MN} > 0$. Since $\text{Hom}_{D^b(\text{rep}^+(Q))}(X[1], N) = 0$ for any representation X

in $\text{rep}^+(Q)$, we have $\text{rad}_{\text{rep}^+(Q)}^2(M, N) = \text{rad}_{D^b(\text{rep}^+(Q))}^2(M, N)$, and consequently, $\text{irr}_{D^b(\text{rep}^+(Q))}(M, N) = \text{irr}_{\text{rep}^+(Q)}(M, N)$. This yields

$$\begin{aligned} \dim_{k_M} \text{irr}_{D^b(\text{rep}^+(Q))}(M, N) &= \dim_{k_M} \text{irr}_{\text{rep}^+(Q)}(M, N) \\ &= \dim_{k_N} \text{irr}_{\text{rep}^+(Q)}(M, N) \\ &= \dim_{k_N} \text{irr}_{D^b(\text{rep}^+(Q))}(M, N), \end{aligned}$$

from which we see that the valued arrow $M \rightarrow N$ in $\Gamma_{D^b(\text{rep}^+(Q))}$ has a symmetric valuation (d_{MN}, d_{MN}) , and it is replaced by the d_{MN} unvalued arrows in $\Gamma_{\text{rep}^+(Q)}$ from M to N .

(2) Let $x \in Q^+$ and $y \in Q_0$ be such that the number of arrows in Q from x to y is $d_{xy} > 0$. Then P_y is a direct summand of $\text{rad}P_x$. By Lemma 7.6(3), $\Gamma_{D^b(\text{rep}^+(Q))}$ has a valued arrow $\alpha_{xy} : I_x \rightarrow P_y[1]$. Since $k_{I_x} \cong k \cong k_{P_y[1]}$ by Proposition 1.3, we see that α_{xy} has a symmetric valuation (d, d) . On the other hand, d_{xy} is the maximal integer such that $P_y^{d_{xy}}$ is a direct summand of $\text{rad}P_x$. By Lemma 7.6(3), d_{xy} is the maximal integer such that $D^b(\text{rep}^+(Q))$ has an irreducible morphism $\zeta : I_y \rightarrow (P_y^{d_{xy}})[1]$. Hence $d = d_{xy}$. Therefore, the valued arrow $\alpha_{xy} : I_x \rightarrow P_y[1]$ is replaced by d_{xy} unvalued arrows from I_x to $P_y[1]$, which are indexed by the arrows in Q from x to y .

Finally, every arrow in $\Gamma_{D^b(\text{rep}^+(Q))}$ is a shift of an arrow γ which is of the form $M \rightarrow N$ or $M \rightarrow N[1]$, where $M, N \in \Gamma_{\text{rep}^+(Q)}$. By Lemma 7.6, γ is as stated in (1) or (2). The proof of the lemma is completed.

As an immediate consequence of Theorem 7.5, the following result describes the Auslander-Reiten translation of $\Gamma_{D^b(\text{rep}^+(Q))}$ which we write as τ_D .

7.8. LEMMA. *If M is a representation lying in $\Gamma_{\text{rep}^+(Q)}$, then*

- (1) $\tau_D M$ is defined if and only if either τM is defined with $\tau_D M = \tau M$, or $M = P_x$ for some $x \in Q^+$ with $\tau_D M = I_x[-1]$;
- (2) $\tau_D^- M$ is defined if and only if either $\tau^- M$ is defined with $\tau_D^- M = \tau^- M$ or $M = I_x$ for some $x \in Q^+$ with $\tau_D^- M = P_x[1]$.

If Q is connected, then $\Gamma_{\text{rep}^+(Q)}$ has a unique preprojective component \mathcal{P}_Q . By Lemmas 7.7 and 7.8, $\Gamma_{D^b(\text{rep}^+(Q))}$ has a connected component \mathcal{C}_Q which is obtained by gluing \mathcal{P}_Q together with the shifts by -1 of the preinjective components of $\Gamma_{\text{rep}^+(Q)}$ in the following way: for each pair (x, y) with $x \in Q^+$ and y an immediate successor of x in Q , one draws d_{xy} arrows from $I_x[-1]$ to P_y , where d_{xy} is the number of arrows from x to y in Q . We call \mathcal{C}_Q the *connecting component* of $\Gamma_{D^b(\text{rep}^+(Q))}$ and describe its shape in the following result.

7.9. PROPOSITION. *Let Q be a connected strongly locally finite quiver. The connecting component \mathcal{C}_Q of $\Gamma_{D^b(\text{rep}^+(Q))}$ embeds in $\mathbb{Z}Q^{\text{op}}$. Furthermore,*

- (1) *If Q has no infinite path, then \mathcal{C}_Q is of shape $\mathbb{Z}Q^{\text{op}}$.*
- (2) *If Q has right infinite but no left infinite paths, then \mathcal{C}_Q is of shape $\mathbb{N}^- \Delta$ with Δ the right-most section of the preprojective component of $\Gamma_{\text{rep}^+(Q)}$.*
- (3) *If Q has left infinite but no right infinite paths, then \mathcal{C}_Q is isomorphic to a right stable translation subquiver of $\mathbb{Z}Q^{\text{op}}$.*

Proof. By Theorem 4.6, the preprojective component \mathcal{P}_Q of $\Gamma_{\text{rep}^+(Q)}$ has a left-most section P_Q generated by the P_x with $x \in Q_0$ and isomorphic to Q^{op} . If M is a

preinjective representation in $\Gamma_{\text{rep}^+(Q)}$, then it lies in the τ -orbit of some injective representation I_y with $y \in Q^+$. By Lemma 7.8(2), $M[-1]$ lies in the τ_D -orbit of P_y . This shows that P_Q is a section of \mathcal{C}_Q , and consequently, \mathcal{C}_Q embeds in $\mathbb{Z}Q^{\text{op}}$.

(1) Suppose that Q has no infinite path. By Theorem 4.7, $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component \mathcal{I} of shape $\mathbb{N}^-Q^{\text{op}}$, while \mathcal{P}_Q is of shape $\mathbb{N}Q^{\text{op}}$ by Theorem 4.6(1). Therefore, $\mathcal{C}_Q \cong \mathbb{Z}Q^{\text{op}}$.

(2) Suppose that Q has right infinite but no left infinite paths. By Theorem 4.7, $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component \mathcal{I}_Q of shape $\mathbb{N}^-Q^{\text{op}}$, and \mathcal{P}_Q has a right-most section Δ by Theorem 4.6(2). Hence \mathcal{C}_Q is of shape $\mathbb{N}^- \Delta$.

(3) Suppose that Q has left infinite but no right infinite paths. By Theorem 4.6, the P_x with $x \in Q_0$ are right stable in $\Gamma_{\text{rep}^+(Q)}$, and hence right stable in $\Gamma_{D^b(\text{rep}^+(Q))}$. Containing a section of right stable vertices, \mathcal{C}_Q is right stable as a translation quiver. Thus, \mathcal{C}_Q is isomorphic to a right stable translation subquiver of $\mathbb{Z}Q^{\text{op}}$. The proof of the theorem is completed.

By Lemmas 7.7 and 7.8, a regular component of $\Gamma_{\text{rep}^+(Q)}$ remains to be a connected component of $\Gamma_{D^b(\text{rep}^+(Q))}$. This yields immediately the following result.

7.10. THEOREM. *If Q is a connected strongly locally finite quiver, then the connected components of $\Gamma_{D^b(\text{rep}^+(Q))}$ are the shifts of the regular components of $\Gamma_{\text{rep}^+(Q)}$ and those of the connecting component \mathcal{C}_Q .*

REMARK. The shapes of the connected components of $\Gamma_{D^b(\text{rep}^+(Q))}$ are described by Theorem 4.14 and Proposition 7.9.

EXAMPLE. Let Q be the quiver as follows :

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ \cdots & \longrightarrow & -2 & \longrightarrow & -1 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & \cdots \end{array}$$

Since the representations P_x with $x \in Q_0$ are all infinite dimensional, the preprojective component of $\Gamma_{\text{rep}^+(Q)}$ is of shape Q^{op} . On the other hand, since $Q^+ = \{0\}$, we see that $\Gamma_{\text{rep}^+(Q)}$ has a unique preinjective component $\{I_0\}$. Therefore, the connecting component of $\Gamma_{D^b(\text{rep}^+(Q))}$ is as follows :

$$\begin{array}{ccccccc} & & & I_0[-1] & \longleftarrow & \cdots & \longleftarrow & P_0 \\ & & & \searrow & & & \nearrow & \\ \cdots & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_{-1} & \longrightarrow & P_{-2} & \longrightarrow & \cdots \end{array}$$

Moreover, by Theorem 6.6(4), $\Gamma_{\text{rep}^+(Q)}$ has infinitely many regular components of wing type, and so does $\Gamma_{D^b(\text{rep}^+(Q))}$.

In case Q is finite, Happel's result says that $D^b(\text{rep}^+(Q))$ has almost split triangles; see [21]. In the infinite case, we are able to find the precise conditions on Q such that $D^b(\text{rep}^+(Q))$ has (left, right) almost split triangles.

7.11. THEOREM. *If Q is a strongly locally finite quiver, then $D^b(\text{rep}^+(Q))$ has (left, right) almost split triangles if and only if Q has no (right, left) infinite path.*

Proof. First, we show that $D^b(\text{rep}^+(Q))$ has left almost split triangles if and only if Q has no right infinite path. Indeed, the necessity follows immediately from Corollary 7.3(1) and Theorem 3.7(1). Suppose that Q has no right infinite path. Then $\text{rep}^+(Q) = \text{rep}^b(Q)$. By Theorem 3.7(1), $\text{rep}^+(Q)$ is left Auslander-Reiten. If I is an injective object in $\text{rep}^+(Q)$, by Proposition 1.16(2), $I = I_x$ for some $x \in Q_0$. Then the socle of I is S_x which has a projective cover P_x in $\text{rep}^+(Q)$. By Corollary 7.3(1), $D^b(\text{rep}^+(Q))$ has left almost split triangles.

Next, we claim that $D^b(\text{rep}^+(Q))$ has right almost split triangles if and only if Q has no left infinite path. Indeed, by Corollary 7.3(2), $D^b(\text{rep}^+(Q))$ has right almost split triangles if and only if $\text{rep}^+(Q)$ is right Auslander-Reiten and every simple representation S_x with $x \in Q_0$ has an injective hull in $\text{rep}^+(Q)$, where the second condition is equivalent to $Q = Q^+$ by Proposition 1.16(3), that is, Q has no left infinite path. Now the claim follows from Theorem 3.7(2).

Finally, it follows from the above two statements that $D^b(\text{rep}^+(Q))$ has almost split triangles if and only if Q has no infinite path. The proof of the theorem is completed.

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RAYMUNDO BAUTISTA, CENTRO DE CIENCIAS MATEMATICAS, UNAM, APARTADO POSTAL 61-3, 58089 MORELIA, MEXICO

E-mail address: `raymundo@matmor.unam.mx`

SHIPING LIU, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, CANADA, J1K 2R1

E-mail address: `shiping.liu@usherbrooke.ca`

CHARLES PAQUETTE, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, CANADA, J1K 2R1

E-mail address: `charles.paquette@usherbrooke.ca`