ALMOST REGULAR AUSLANDER-REITEN COMPONENTS AND QUASITILTED ALGEBRAS

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Introduction

The problem of giving a general description of the shapes of Auslander-Reiten components of an artin algebra has been settled for semiregular components (see [**4, 9, 14**]). Recently, S. Li has considered this problem for components in which every possible path from an injective module to a projective module is sectional. The result says that such a component is embeddable in some $\mathbb{Z}\Delta$ with Δ a quiver without oriented cycles if it contains no oriented cycle. In this note, we shall show that such a component is a semiregular tube if it contains an oriented cycle. In this way, one obtains a complete description of the shapes of such components. For this reason, we propose to call such components *almost regular*. We shall further give some new characterizations of tilted and quasi-tilted algebras (see (2.1), (2.2)), which shows that every Auslander-Reiten component of a quasitilted algebra is almost regular. As an easy application, we shall obtain a result of Coelho-Skowroński [**3**] saying that a quasitilted algebra is tilted if it admits a non-semiregular Auslander-Reiten component.

1. *Almost regular components*

Throughout this note, let A be a connected artin algebra, $\operatorname{mod} A$ be the category of finitely generated right A-modules and $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ generated by the indecomposable modules. We denote by Γ_A the Auslander-Reiten quiver of A and by τ, τ^- the Auslander-Reiten translations DTr, TrD respectively. We shall identify a module X in $\operatorname{ind} A$ with the corresponding vertex $[X]$ (that is, the isomorphism class of X) in Γ_A. We shall say that a module $X \in \Gamma_A$ is *left stable* (respectively, *right stable*) if $\tau^n X$ (respectively, $\tau^{-n} X$) is nonzero for all positive integers n.

Recall that a connected component of Γ_A is *regular* if it contains no projective or injective module; and *semiregular* if it does not contain both a projective module and an injective module.

1.1. Definition. A connected component $\mathcal{C}$ of Γ_A is said to be *almost regular* if every possible path

$$X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$$

in $\mathcal{C}$ with X_0 being injective and X_n being projective is sectional, that is, there is no i with $0 < i < n$ such that $\tau X_{i+1} = X_{i-1}$.

Note that a semiregular Auslander-Reiten component is almost regular by definition. Conversely we have the following result.

1.2. THEOREM. *Let $\mathcal{C}$ be an almost regular component of Γ_A. If $\mathcal{C}$ contains an oriented cycle, then it is semiregular.*

Proof. Assume that $\mathcal{C}$ contains both a projective module and an injective module. We first show that $\mathcal{C}$ contains no τ-periodic module. In fact if this is not true, then $\mathcal{C}$ contains an arrow $M \to N$ or $N \to M$ with M being τ-periodic and N being neither left stable nor right stable. Thus M admits a projective successor P and an injective predecessor I in $\mathcal{C}$. This gives rise to a nonsectional path in $\mathcal{C}$ from I to P, and hence a contradiction. Let now

$$X_1 \to X_2 \to \cdots \to X_{r-1} \to X_r = X_1$$

be an oriented cycle in $\mathcal{C}$. If the X_i are all stable, then they are all τ-periodic [**8**, (2.7)], which is a contradiction. Thus the cycle contains a nonstable module. We need only to consider the case where one of the X_i is not right stable. By applying τ^- if necessary, we may assume that X_1 is injective. Let

$$Y_1 \to Y_2 \to \cdots \to Y_s \to Y_{s+1} \quad (*)$$

be a path in $\mathcal{C}$ of minimal positive length such that Y_1 has an injective predecessor in $\mathcal{C}$ and $Y_{s+1} = \tau^t Y_1$ with $t \geq 0$.

Assume that $t = 0$, that is $Y_{s+1} = Y_1$. Then the path

$$Y_1 \to Y_2 \to \cdots \to Y_s \to Y_{s+1} \to Y_{s+2} = Y_2$$

is not sectional [**1**]. Thus $s > 2$ since Y_1 and Y_2 are not τ-periodic. We shall obtain a contradiction to the minimality of the length of $(*)$ by finding a shorter path of this kind. Let $1 < i_0 < s + 2$ be such that $Y_{i_0-1} = \tau Y_{i_0+1}$. Note that Y_{i_0+1} admits no projective successor in $\mathcal{C}$ since Y_1 has an injective predecessor in $\mathcal{C}$. In particular the Y_i with $1 \leq i \leq s$ are all nonprojective. If $i_0 = s + 1$, then $Y_s = \tau Y_2$ and we have a desired path $Y_2 \to \cdots \to Y_s = \tau Y_2$. If $i_0 = s$, then $Y_{s-1} = \tau Y_{s+1} = \tau Y_1$ and we get a path $Y_1 \to \cdots \to Y_{s-1} = \tau Y_1$. If $1 < i_0 < s$, then

$$Y_1 \to \cdots \to Y_{i_0-1} \to \tau Y_{i_0+2} \to \cdots \to \tau Y_{s+1} = \tau Y_1$$

is a desired path since the Y_i are all nonprojective.

Thus $t > 0$. This implies that Y_1 has no projective successor in $\mathcal{C}$ since $\tau^t Y_1$ has an injective predecessor in $\mathcal{C}$. Suppose that $0 \leq j < t$ and $\mathcal{C}$ contains a path

$$\tau^j Y_1 \to \tau^j Y_2 \to \cdots \to \tau^j Y_s \to \tau^{t+j} Y_1.$$

Since $j < t$, the module $\tau^j Y_1$ is a successor of $\tau^t Y_1$, and hence of Y_1 in $\mathcal{C}$. Thus $\tau^{t+j} Y_1$ and the $\tau^j Y_i$ with $1 \leq i \leq s$ are all nonprojective. Thus $\mathcal{C}$ contains a path

$$\tau^{j+1} Y_1 \to \tau^{j+1} Y_2 \to \cdots \to \tau^{j+1} Y_s \to \tau^{t+j+1} Y_1.$$

By induction, $\mathcal{C}$ contains a path

$$\tau^t Y_1 \to \tau^t Y_2 \to \cdots \to \tau^t Y_s \to \tau^{2t} Y_1.$$

Continuing this argument, we conclude that for all $i \geq 0$, $\mathcal{C}$ contains a path

$$\tau^{it} Y_1 \to \tau^{it} Y_2 \to \cdots \to \tau^{it} Y_s \to \tau^{(i+1)t} Y_1.$$

Therefore the Y_i with $1 \leq i \leq s$ are all left stable, and $\mathcal{C}$ contains an infinite path

$$Y_1 \to Y_2 \to \cdots \to Y_s \to \tau^t Y_1 \to \tau^t Y_2 \to \cdots \to \tau^t Y_s \to \tau^{2t} Y_1 \to \cdots \qquad (**)$$

Suppose that $Y_j = \tau^k Y_i$ with $1 \leq i < j \leq s$ and $k \in \mathbb{Z}$. Then we have two paths $Y_i \to \cdots \to Y_j = \tau^k Y_i$ and

$$Y_j \to \cdots \to Y_s \to \tau^t Y_1 \to \cdots \to \tau^t Y_i = \tau^{t-k} Y_j$$

of length less than s. This is again a contradiction to the minimality of the length of $(*)$ since either $k \geq 0$ or $t - k \geq 0$. Therefore the Y_i with $1 \leq i \leq s$ pairwise belong to different τ-orbits. In particular the infinite path $(**)$ is sectional.

Let Γ be the left stable component of Γ_A containing the Y_i. Then Γ contains oriented cycles but no τ-periodic module. Thus every module in Γ admits at most two immediate predecessors in Γ [**9**, (2.3)]. We shall prove that the Y_i with $1 \leq i \leq s$ meet each τ-orbit of Γ. Indeed, let $\tau^j Y_i$ with $1 \leq i \leq s$ and $j \in \mathbb{Z}$ be a module in Γ and Z an immediate successor of $\tau^j Y_i$ in Γ. Let p, q be positive integers such that $p + j = qt$. Then $\tau^{p+1} Z$ is an immediate predecessor of $\tau^{p+j} Y_i = \tau^{qt} Y_i$ in Γ. Since $q > 0$, the module $\tau^{qt} Y_i$ has two distinct immediate predecessors in Γ which lie in the τ-orbit of the Y_i with $1 \leq i \leq s$. Therefore Z lies in the τ-orbit of the Y_i with $1 \leq i \leq s$.

Let $U = \tau^n Y_i$ with $1 \leq i \leq s$ and $n \in \mathbb{Z}$ be a module in Γ. If $n \leq 0$, then U is clearly a successor of Y_1 in Γ. If $n > 0$, then $n = td + m$ with $d \geq 0$ and $0 \leq m < t$. Therefore U is a successor of $\tau^{(d+1)t} Y_i$, and hence of Y_1 in Γ. This shows that every module in Γ is a successor of Y_1 in Γ. Suppose that Γ is different from $\mathcal{C}$, that is $\mathcal{C}$ contains a projective module. Then $\mathcal{C}$ contains an arrow $M \to P$ with $M \in \Gamma$ and P being projective. Thus P is a successor of Y_1 in $\mathcal{C}$, which is a contradiction. Therefore $\mathcal{C} = \Gamma$ is left stable. This completes the proof the theorem.

Let $\mathcal{C}$ be a connected component of Γ_A. Recall that a *section* of $\mathcal{C}$ is a connected full convex subquiver which contains no oriented cycle and meets exactly once each τ-orbit of $\mathcal{C}$ (see [**11**, section 2]). The main result of [**7**] says that $\mathcal{C}$ contains a section Δ if and only if $\mathcal{C}$ is almost regular and contains no oriented cycle. In this case, $\mathcal{C}$ can be embedded in $\mathbb{Z}\Delta$ [**9**, (3.2)]. Combining these results with those in [**4**], [**9**, (2.5)] and [**14**], we obtain the following description of the shapes of almost regular Auslander-Reiten components.

1.3. THEOREM. *Let $\mathcal{C}$ be an almost regular component of Γ_A. Then $\mathcal{C}$ is either a ray tube, a coray tube, a stable tube or can be embedded in some $\mathbb{Z}\Delta$ with Δ a valued quiver without oriented cycles.*

We conclude this section by studying some behaviors of the maps involving modules from an Auslander-Reiten component containing a section. Recall that a *path* in $\operatorname{ind} A$ is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$$

of nonzero non-isomorphisms in $\operatorname{ind} A$. In this case, we call X_0 a *predecessor* of X_n, and X_n a *successor* of X_0 in $\operatorname{ind} A$. Moreover the path is said to be *sectional* if there is no i with $0 < i < n$ such that $\tau X_{i+1} \cong X_{i-1}$. Thus a (sectional) path of irreducibles maps in $\operatorname{ind} A$ gives rise to a (sectional) path in Γ_A and vice versa.

1.4. LEMMA. *Let $\mathcal{C}$ be a connected component of Γ_A containing a section Δ. Let $f : X \to Y$ be a nonzero map in* $\operatorname{ind} A$. *If Y lies in some $\tau^r \Delta$ with $r \in \mathbb{Z}$ while X is not a predecessor of Y in $\mathcal{C}$, then $\tau^n \Delta$ with $n \geq r$ contains a module which is a successor of X in* $\operatorname{ind} A$.

Proof. Assume that Y lies in $\tau^r \Delta$ and X is not a predecessor of Y in $\mathcal{C}$. We shall use induction on $s = n - r$. The lemma is trivially true for $s = 0$. Suppose that $s > 0$ and the lemma is true for $s - 1$. Since X is not a predecessor of Y in $\mathcal{C}$ and f is nonzero, there is an infinite path

$$\cdots \to Y_i \to Y_{i-1} \to \cdots \to Y_1 \to Y_0 = Y$$

in $\mathcal{C}$ such that $\operatorname{Hom}_A(X, Y_i) \neq 0$ for all $i \geq 0$. Since $\mathcal{C}$ is embedded in $\mathbb{Z}\Delta$, every Y_i belongs to some $\tau^{r_i} \Delta$ with $r_i \geq r$. Now the lemma is true for s if there is some $r_i \geq n$. Otherwise, there is some $i_0 \geq 0$ such that $r_i = r_{i_0}$ for all $i \geq i_0$. Therefore the path

$$\cdots \to Y_j \to Y_{j-1} \to \cdots \to Y_{i_0+1} \to Y_{i_0}$$

lies entirely in $\tau^{r_{i_0}} \Delta$, and hence is sectional. By Lemma 2 of [**S**], there is some $p, q \geq r_0$ such that $\operatorname{Hom}_A(Y_p, \tau Y_q) \neq 0$. Note that Y_p is not a predecessor of τY_q in $\mathcal{C}$. By inductive hypothesis, there is a module in $\tau^n \Delta$ which is a successor of Y_p, and hence of X in $\operatorname{ind} A$. The proof is completed.

2. *Quasitilted algebras*

We begin this section with a new characterization of tilted algebras which shows the separating property of a complete slice. We denote by $D(A)$ the standard injective cogenerator of $\operatorname{mod} A$.

2.1. THEOREM. *Let $\mathcal{C}$ be a connected component of Γ_A. Then A is tilted with $\mathcal{C}$ a connecting component of Γ_A if and only if $\mathcal{C}$ contains a section Δ satisfying*:

(1) $\operatorname{Hom}_A(X, \tau Y) = 0$ *for all* $X, Y \in \Delta$,

(2) $\operatorname{Hom}_A(\tau^- X, A) = 0$ *for all* $X \in \Delta$, *and*

(3) $\mathrm{Hom}_A(D(A), \tau X) = 0$ *for all* $X \in \Delta$.

Proof. Assume that A is tilted and $\mathcal{C}$ is a connecting component of Γ_A. Let $\mathcal{S}$ be a complete slice in $\mathrm{mod}\, A$ whose indecomposable objects lie in $\mathcal{C}$. It is then well-known that the full subquiver Δ of $\mathcal{C}$ generated by the indecomposable objects of $\mathcal{S}$ is a desired section of $\mathcal{C}$.

Conversely let Δ be a section of $\mathcal{C}$ satisfying the conditions stated in the theorem. Then Δ is finite [**13**, Lemma 2]. Let T be the direct sum of the modules in Δ. Then T is a partial tilting module of injective dimension less than two (see, for example, [**12**, (2.4)]). Hence there is a module N in mod-A such that $T \oplus N$ is tilting module [**2**, (2.1)]. Assume that there is an indecomposable direct summand U of N that is not a direct summand of T. Then either $\mathrm{Hom}_A(U, T) \neq 0$ or $\mathrm{Hom}_A(T, U) \neq 0$ since $\mathrm{End}_A(T \oplus N)$ is connected. This implies that either $\mathrm{Hom}_A(U, \tau T) \neq 0$ or $\mathrm{Hom}_A(\tau^- T, U) \neq 0$ since Δ is a finite section of $\mathcal{C}$. Therefore either $\mathrm{Ext}^1_A(T, U) \neq 0$ or $\mathrm{Ext}^1_A(U, T) \neq 0$. This is contrary to $T \oplus N$ being a tilting module. Therefore T is a tilting module, and hence a faithful module. It follows now from [**10**, (1.6)] that A is tilted and $\mathcal{C}$ is a connecting component of Γ_A.

Recall that A is *quasitilted* if the global dimension of A is at most two and every module in $\mathrm{ind}\, A$ is either of projective dimension less than two or of injective dimension dimension less than two. There are many characterizations of quasitilted algebras (see [**5**]). We note that the following is convenient in certain cases.

2.2. Proposition. *An artin algebra A is quasitilted if and only if every possible path in* $\mathrm{ind}\, A$ *from an injective module to a projective module is sectional.*

Proof. We first give the proof of sufficiency which is due to Happel. Assume that A is not quasitilted. If the global dimension of A is greater than two, then there is a simple A-module S of projective dimension greater than two. Hence the first syzygy of S has an indecomposable direct summand X of projective dimension greater than one. Therefore $\mathrm{Hom}_A(D(A), \tau X) \neq 0$. Note that X is a submodule of the radical of the projective cover of S. This gives rise to a nonsectional path in $\mathrm{ind}\, A$ from an injective module to a projective module. If there is some Y in $\mathrm{ind}\, A$ of projective and injective dimensions both greater than one, then $\mathrm{Hom}_A(D(A), \tau X) \neq 0$ and $\mathrm{Hom}_A(\tau^- X, A) \neq 0$. So we can also find a nonsectional path in $\mathrm{ind}\, A$ from an injective module to a projective module.

Assume now that

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$$

is a nonsectional path in $\mathrm{ind}\, A$ with X_0 being injective and X_n being projective. We shall show that X_n has a predecessor in $\mathrm{ind}\, A$ whose projective dimension is greater than one. This implies that A is not quasitilted [**5**, (1.14)]. Indeed, let $0 < r < n$ be such that $\tau X_{r+1} = X_{r-1}$. Then $r > 1$ since X_0 is injective. If $f_1 \cdots f_{r-1} \neq 0$, then the projective dimension of X_{r+1} is greater than one. Suppose that $f_1 \cdots f_r = 0$. Let $1 < s \leq r$ be such that $f_1 \cdots f_{s-1} \neq 0$ and

$(f_1 \cdots f_{s-1})f_s = 0$. By the lemma of Section 1 of [**6**], there is some Z in $\operatorname{ind} A$ such that $\operatorname{Hom}_A(X_0, \tau Z) \neq 0$ and $\operatorname{Hom}_A(Z, X_s) \neq 0$. Therefore Z is a predecessor of X_n in $\operatorname{ind} A$ of projective dimension greater than one. The proof is completed.

As an immediate consequence, every connected component of the Auslander-Reiten quiver of a quasitilted algebra is almost regular (see also [**5**, (1.11)]).

2.3. THEOREM [**3**]. *Let A be a connected quasitilted artin algebra. If Γ_A contains a non-semiregular component $\mathcal{C}$, then A is tilted with $\mathcal{C}$ the connecting component of Γ_A.*

Proof. Let $\mathcal{C}$ be a non-semiregular component of Γ_A. By Theorem 1.2, $\mathcal{C}$ contains no oriented cycle. By [**7**, (2.10)], $\mathcal{C}$ contains a section Δ such that every module in Δ has an injective predecessor in Δ while $\tau\Delta$ has no injective predecessor in $\mathcal{C}$. Dually $\mathcal{C}$ contains a section Δ_1 such that every module in Δ_1 has a projective successor in Δ_1.

Assume that $\operatorname{Hom}_A(\tau^- X, P) \neq 0$ with $X \in \Delta$ and $P \in \operatorname{ind} A$ being projective. Since X has an injective predecessor I in Δ, we have a nonsectional path in $\operatorname{ind} A$ from I to P, which is a contradiction. Suppose that there is a path in $\operatorname{ind} A$ from an injective module I_0 to a module τX with $X \in \Delta$. Then I_0 is not a predecessor of τX in $\mathcal{C}$. Applying Lemma 1.4 to Δ_1, we get a module $Y \in \Delta_1$ such that τY is a successor of I_0 in $\operatorname{ind} A$. Since Y admits a projective successor P_0 in Δ_1, this gives rise to a nonsectional path in $\operatorname{ind} A$ from I_0 to P_0, which is impossible. Therefore there is no module in $\tau\Delta$ which is a successor of an injective module in $\operatorname{ind} A$. In particular $\operatorname{Hom}_A(D(A), \tau\Delta) = 0$ and $\operatorname{Hom}_A(\Delta, \tau\Delta) = 0$. By Theorem 2.1, A is tilted and $\mathcal{C}$ is the connecting component of Γ_A. This completes the proof.

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